

PLMP - Point-Line Minimal Problems in Complete Multi-View Visibility

Kathlén Kohn

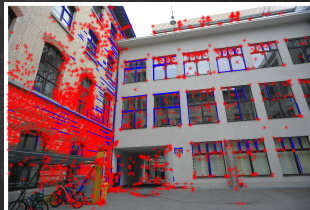
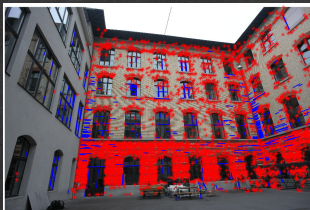
joint work with Timothy Duff, Anton Leykin & Tomas Pajdla

March 29, 2019

Reconstruct 3D scenes and camera poses
from 2D images

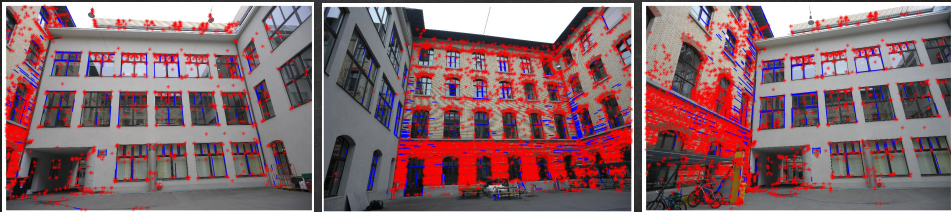
Reconstruct 3D scenes and camera poses from 2D images

- ◆ Step 1: Identify common points and lines on given images



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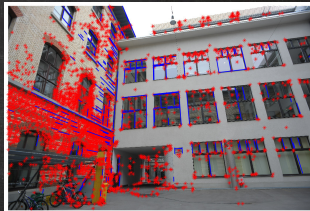
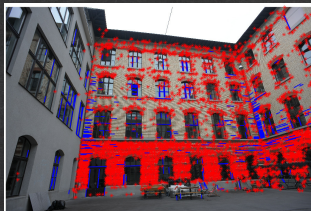
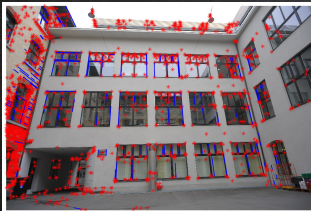
- ◆ Step 1: Identify common points and lines on given images



- ◆ Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

Reconstruct 3D scenes and camera poses from 2D images

- ◆ Step 1: Identify common points and lines on given images



- ◆ Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.



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This problem has 20 solutions over \mathbb{C} .

(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

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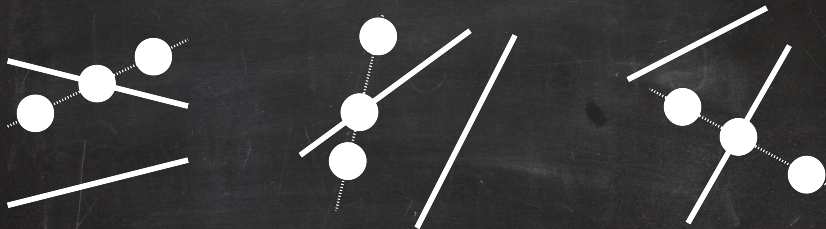
This problem has **20** solutions over \mathbb{C} .

(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

⇒ The 5-Point-Problem is a **minimal** problem!

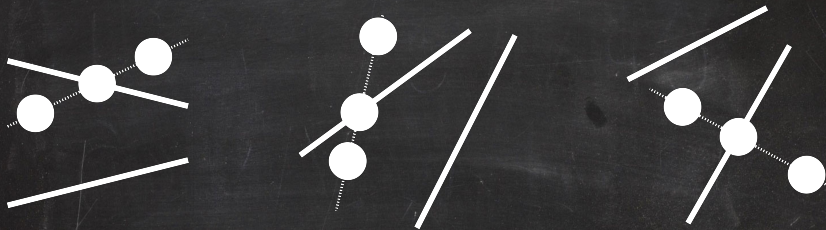
Another minimal problem

- ◆ Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- ◆ Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses



Another minimal problem

- ◆ Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- ◆ Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses



This problem has **40** solutions over \mathbb{C} .

(solution = 3 camera poses and 3D coordinates of points and lines)

⇒ It is a **minimal** problem!

Minimal Problems

A **Point-Line-Problem (PLP)** consists of

- ◆ a number m of cameras,
- ◆ a number p of points,
- ◆ a number ℓ of lines,
- ◆ a set \mathcal{I} of incidences between points and lines.

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Definition

A PLP is **minimal** if,

given m random images of p points and ℓ lines with incidences \mathcal{I} ,
it has a positive and finite number of solutions over \mathbb{C} .

(solution = m camera poses and 3D coordinates of points and lines)

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(solution = m camera poses and 3D coordinates of points and lines)

Can we list **all** minimal PLPs?
How many solutions do they have?

Minimal PLPs

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^f p^{d,l} l_a^a$	1021 ₁	1013 ₃	1005 ₅	2011 ₁	2003 ₂	2003 ₃	1030 ₀	1022 ₂	1014 ₄	1006 ₆	3001 ₁	2110 ₀	2102 ₁
(p, l, \mathcal{I})													
Minimal Degree	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
	> 450k*			11306*	26240*	11008*	3040*	4524*			1728*	32*	544*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^f p^{d,l} l_a^a$	2102 ₂	1040 ₀	1032 ₂	1024 ₄	1016 ₆	1008 ₈	2021 ₁	2013 ₂	2013 ₃	2005 ₃	2005 ₄	2005 ₅	3010 ₀
(p, l, \mathcal{I})													
Minimal Degree	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
	544*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^f p^{d,l} l_a^a$	3002 ₁	3002 ₂	2111 ₁	2103 ₁	2103 ₂	2103 ₃	3100 ₀	2201 ₁	5000 ₂	4100 ₃	3200 ₃	3200 ₄	2300 ₅
(p, l, \mathcal{I})													
Degree	312	224	40	144	144	144	64		20	16	12		

Joint camera map

(3D-arrangement, $\text{cam}_1, \dots, \text{cam}_m$)
of p points and ℓ lines
with incidences \mathcal{I}

Joint camera map

(3D-arrangement of p points and ℓ lines with incidences \mathcal{I} , $\text{cam}_1, \dots, \text{cam}_m$) \mapsto (2D-arr₁, ..., 2D-arr_m)

Joint camera map

$$\begin{array}{ccccc}
 X & \times & C & \longrightarrow & Y \\
 \text{(3D-arrangement} & , & \text{cam}_1, \dots, \text{cam}_m) & \longmapsto & \text{(2D-arr}_1, \dots, \text{2D-arr}_m) \\
 \text{of } p \text{ points and } \ell \text{ lines} & & & & \\
 \text{with incidences } \mathcal{I} & & & &
 \end{array}$$

♦ $X = \{ \text{3D-arr. of } p \text{ points and } \ell \text{ with incidences } \mathcal{I} \}$

Joint camera map

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- ◆ $X = \{ \text{3D-arr. of } p \text{ points and } \ell \text{ with incidences } \mathcal{I} \}$
- ◆ $C = \{ m \text{ camera poses} \}$

Joint camera map

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 \end{array}
 \begin{array}{c}
 \longrightarrow \\
 Y \\
 \text{(2D-arr}_1, \dots, \text{2D-arr}_m)
 \end{array}$$

- ◆ $X = \{ \text{3D-arr. of } p \text{ points and } \ell \text{ with incidences } \mathcal{I} \}$
- ◆ $C = \{ m \text{ camera poses } \}$
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- ◆ $X = \{ \text{3D-arr. of } p \text{ points and } \ell \text{ with incidences } \mathcal{I} \}$
- ◆ $C = \{ m \text{ camera poses} \}$
- ◆ $Y = \{ m \text{ 2D-arr. of } p \text{ points and } \ell \text{ with incidences } \mathcal{I} \}$

Lemma

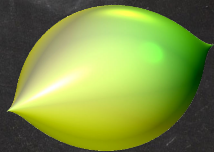
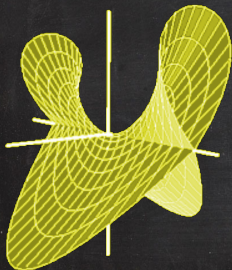
If a PLP is minimal, then $\dim(X) + \dim(C) = \dim(Y)$.

Algebraic varieties

Definition

A **variety** is the common zero set of a system of polynomial equations.

A variety looks like a manifold **almost everywhere**:

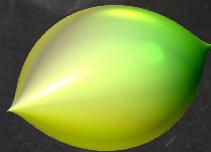
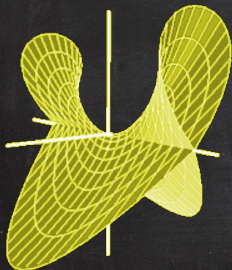


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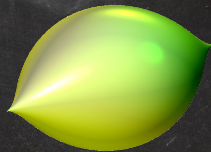
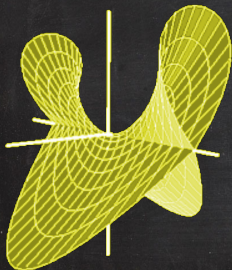
The **dimension** of a variety is its local dimension as a manifold.

Algebraic varieties

Definition

A **variety** is the common zero set of a system of polynomial equations.

A variety looks like a manifold **almost everywhere**:



Definition

The **dimension** of a variety is its local dimension as a manifold.

X , C and Y are varieties!

Deriving the big table

$$\begin{array}{ccccc} X & \times & C & \longrightarrow & Y \\ (3\text{D-arrangement} & , & \text{cam}_1, \dots, \text{cam}_m) & \longmapsto & (2\text{D-arr}_1, \dots, 2\text{D-arr}_m) \\ \text{of } p \text{ points and } \ell \text{ lines} & & & & \\ \text{with incidences } \mathcal{I} & & & & \end{array}$$

Lemma

If a PLP is minimal, then $\dim(X) + \dim(C) = \dim(Y)$.

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Lemma

If a PLP is minimal, then $\dim(X) + \dim(C) = \dim(Y)$.

Theorem

♦ *If $m > 6$, then $\dim(X) + \dim(C) \neq \dim(Y)$.*

Deriving the big table

X \times C \longrightarrow Y
 (3D-arrangement of p points and ℓ lines with incidences \mathcal{I} , $\text{cam}_1, \dots, \text{cam}_m$) \longmapsto $(2\text{D-arr}_1, \dots, 2\text{D-arr}_m)$

Lemma

If a PLP is minimal, then $\dim(X) + \dim(C) = \dim(Y)$.

Theorem

- ♦ If $m > 6$, then $\dim(X) + \dim(C) \neq \dim(Y)$.
- ♦ There are exactly 39 PLPs with $\dim(X) + \dim(C) = \dim(Y)$:

m views	6	6	6	5	5	5	4	4	4	4	4	4	4	4
$p^1 p^0 \ell^1 \ell_0$	1021 ₁	1013 ₃	1005 ₅	2011 ₁	2003 ₂	2003 ₃	1030 ₀	1022 ₂	1014 ₄	1006 ₆	3001 ₁	2110 ₀	2102 ₁	2102 ₁
(p, ℓ, \mathcal{I})														
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y	Y
Degree	> 450k*			11306*	26240*	11008*	3040*	4524*			1728*	32*	544*	

m views	4	3	3	3	3	3	3	3	3	3	3	3	3	3
$p^1 p^0 \ell^1 \ell_0$	2102 ₂	1040 ₀	1032 ₂	1024 ₄	1016 ₆	1008 ₈	2021 ₁	2013 ₂	2013 ₃	2005 ₃	2005 ₄	2005 ₅	3010 ₀	3010 ₀
(p, ℓ, \mathcal{I})														
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y	Y
Degree	544*	360	552	480			264	432	328	480	240	64	216	

m views	3	3	3	3	3	3	3	3	2	2	2	2	2	2
$p^1 p^0 \ell^1 \ell_0$	3002 ₁	3002 ₂	2111 ₁	2103 ₁	2103 ₂	2103 ₃	3100 ₀	2201 ₁	5000 ₂	4100 ₃	3200 ₃	3200 ₄	2300 ₅	2300 ₅
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 X & \times & C & \longrightarrow & Y \\
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 \text{of } p \text{ points and } \ell \text{ lines} & & & & \\
 \text{with incidences } \mathcal{I} & & & &
 \end{array}$$

Lemma

A PLP with $\dim(X) + \dim(C) = \dim(Y)$ is minimal if and only if its joint camera map $X \times C \rightarrow Y$ is dominant.

Deriving the big table

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 X & \times & C \\
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 \end{array}
 \begin{array}{ccc}
 \longrightarrow & & Y \\
 & & (2\text{D-arr}_1, \dots, 2\text{D-arr}_m)
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Definition

A map $\varphi : A \rightarrow B$ is **surjective** if for every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

Definition

A map $\varphi : A \rightarrow B$ is **dominant** if for **almost** every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

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Definition

A map $\varphi : A \rightarrow B$ is **dominant** if for **almost** every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

Fact

A map $\varphi : A \rightarrow B$ between variety A and B is dominant if and only if for almost every $a \in A$ the differential $D_a\varphi : T_aA \rightarrow T_{\varphi(a)}B$ is surjective.

Deriving the big table

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 X & \times & C \\
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 \end{array}
 \begin{array}{c}
 \longrightarrow \\
 Y \\
 (2D\text{-arr}_1, \dots, 2D\text{-arr}_m)
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Can check this computationally! It is only linear algebra!

m views $p^f p^d l^f l_\alpha^a$	6 1021 ₁	6 1013 ₃	6 1005 ₅	5 2011 ₁	5 2003 ₂	5 2003 ₃	4 1030 ₀	4 1022 ₂	4 1014 ₄	4 1006 ₆	4 3001 ₁	4 2110 ₀	4 2102 ₁
(p, l, \mathcal{I})													
Minimal Degree	Y > 450k*	N	N	Y 11306*	Y 26240*	Y 11008*	Y 3040*	Y 4524*	N	N	Y 1728*	Y 32*	Y 544*

m views $p^f p^d l^f l_\alpha^a$	4 2102 ₂	3 1040 ₀	3 1032 ₂	3 1024 ₄	3 1016 ₆	3 1008 ₈	3 2021 ₁	3 2013 ₂	3 2013 ₃	3 2005 ₃	3 2005 ₄	3 2005 ₅	3 3010 ₀
(p, l, \mathcal{I})													
Minimal Degree	Y 544*	Y 360	Y 552	Y 480	N	N	Y 264	Y 432	Y 328	Y 480	Y 240	Y 64	Y 216

m views $p^f p^d l^f l_\alpha^a$	3 3002 ₁	3 3002 ₂	3 2111 ₁	3 2103 ₁	3 2103 ₂	3 2103 ₃	3 3100 ₀	3 2201 ₁	2 5000 ₂	2 4100 ₃	2 3200 ₃	2 3200 ₄	2 2300 ₅
(p, l, \mathcal{I})													
Degree	312	224	40	144	144	144	64		20	16	12		

m views $p^f p^d l^f l_\alpha^a$	6	6	6	5	5	5	4	4	4	4	4	4	4
(p, l, \mathcal{I})													
Minimal Degree	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
	$> 450k^*$			11306*	26240*	11008*	3040*	4524*			1728*	32*	544*

m views $p^f p^d l^f l_\alpha^a$	4	3	3	3	3	3	3	3	3	3	3	3	3
(p, l, \mathcal{I})													
Minimal Degree	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
	544*	360	552	480			264	432	328	480	240	64	216

m views $p^f p^d l^f l_\alpha^a$	3	3	3	3	3	3	3	3	2	2	2	2	2
(p, l, \mathcal{I})													
Degree	312	224	40	144	144	144	64		20	16	12		

- ◆ For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases**
(standard technique in algebraic geometry)

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^f p^{dlf} l_\alpha^a$	1021 ₁	1013 ₃	1005 ₅	2011 ₁	2003 ₂	2003 ₃	1030 ₀	1022 ₂	1014 ₄	1006 ₆	3001 ₁	2110 ₀	2102 ₁
(p, l, \mathcal{I})													
Minimal Degree	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
	> 450k*			11306*	26240*	11008*	3040*	4524*			1728*	32*	544*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^f p^{dlf} l_\alpha^a$	2102 ₂	1040 ₀	1032 ₂	1024 ₄	1016 ₆	1008 ₈	2021 ₁	2013 ₂	2013 ₃	2005 ₃	2005 ₄	2005 ₅	3010 ₀
(p, l, \mathcal{I})													
Minimal Degree	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
	544*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^f p^{dlf} l_\alpha^a$	3002 ₁	3002 ₂	2111 ₁	2103 ₁	2103 ₂	2103 ₃	3100 ₀	2201 ₁	5000 ₂	4100 ₃	3200 ₃	3200 ₄	2300 ₅
(p, l, \mathcal{I})													
Degree	312	224	40	144	144	144	64		20	16	12		

- ◆ For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)
- ◆ For $m \in \{4, 5, 6\}$: compute number of solutions with **homotopy continuation** and **monodromy** (state-of-the-art method in numerical algebraic geometry)