PLMP - Point-Line Minimal Problems in Complete Multi-View Visibility

Kathlén Kohn

joint work with Timothy Duff, Anton Leykin & Tomas Pajdla

March 29, 2019

Reconstruct 3D scenes and camera poses from 2D images

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- Step 1: Identify common points and lines on given images

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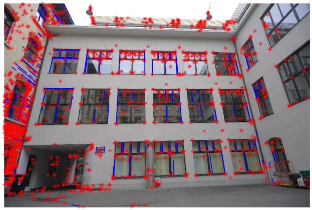

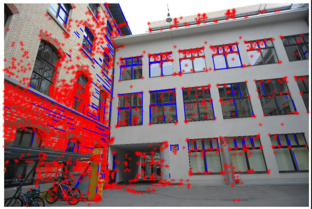

- Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

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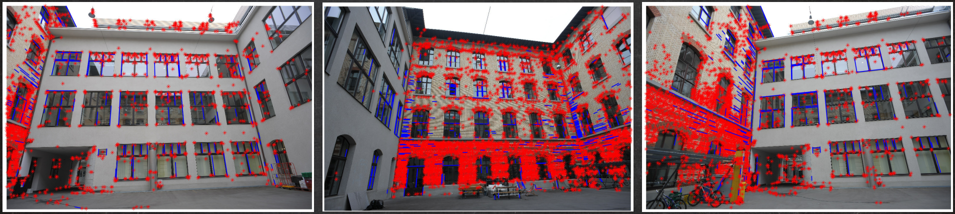

- **Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses**

5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.

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This problem has 20 solutions over $\mathbb{C}$.
(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

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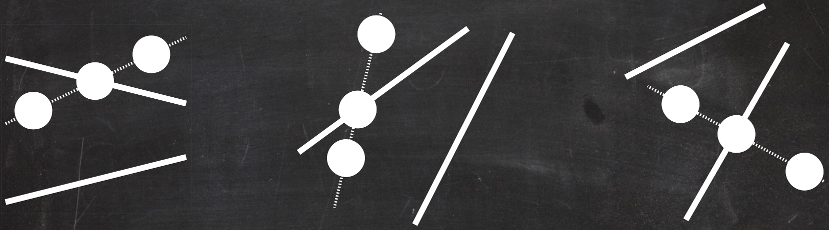
$\Rightarrow$ **The 5-Point-Problem is a minimal problem!**

Another minimal problem

- Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses



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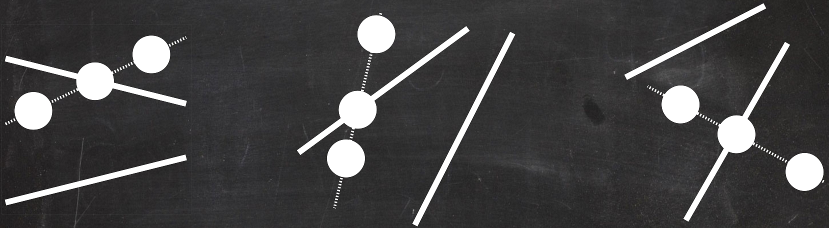

This problem has 40 solutions over $\mathbb{C}$.
(solution = 3 camera poses and 3D coordinates of points and lines)

$\Rightarrow$ **It is a minimal problem!**

Minimal Problems

A **Point-Line-Problem (PLP)** consists of

- a number m of cameras,
- a number p of points,
- a number ℓ of lines,
- a set $\mathcal{I}$ of incidences between points and lines.

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Definition
A PLP is **minimal** if,
given m random images of p points and ℓ lines with incidences $\mathcal{I}$,
it has a positive and finite number of solutions over $\mathbb{C}$.

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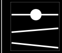
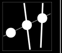
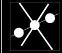
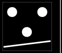
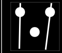
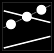
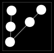
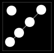
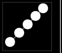
Can we list all minimal PLPs?
How many solutions do they have?

Minimal PLPs

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p,l,\mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p,l,\mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p,l,\mathcal{I})$													
Degree	312	224	40	144	144	144	64		20	16	12		

Joint camera map

(3D-arrangement of p points and ℓ lines with incidences $\mathcal{I}$, $\text{cam}_1, \ldots, \text{cam}_m$)

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$$(\text{3D-arrangement} \quad , \quad \text{cam}_1, \ldots, \text{cam}_m) \quad \longmapsto \quad (\text{2D-arr}_1, \ldots, \text{2D-arr}_m)$$

of p points and ℓ lines
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$$\begin{array}{ccccc} X & \times & C & \longrightarrow & Y \\ \text{(3D-arrangement} & , & \text{cam}_1, \ldots, \text{cam}_m) & \longmapsto & (\text{2D-arr}_1, \ldots, \text{2D-arr}_m) \\ \text{of } p \text{ points and } \ell \text{ lines} & & & & \\ \text{with incidences } \mathcal{I} & & & & \end{array}$$

- $X = \{$ 3D-arr. of p points and ℓ with incidences $\mathcal{I}\}$

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Lemma

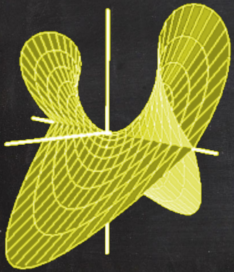
If a PLP is minimal, then $\dim(X) + \dim(C) = \dim(Y)$.

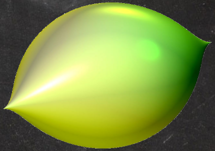
Algebraic varieties

Definition

A **variety** is the common zero set of a system of polynomial equations.

A variety looks like a manifold **almost everywhere**:





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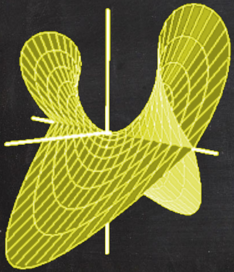

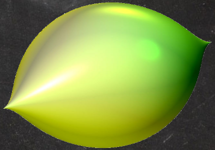

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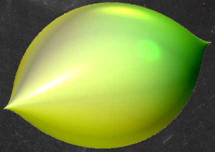

Definition
The **dimension** of a variety is its local dimension as a manifold.

***X*, *C* and *Y* are varieties!**

Deriving the big table

$$\begin{array}{ccccc} X & \times & C & \longrightarrow & Y \\ (\text{3D-arrangement} & , & \text{cam}_1, \ldots, \text{cam}_m) & \longmapsto & (\text{2D-arr}_1, \ldots, \text{2D-arr}_m) \end{array}$$

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If a PLP is minimal, then $\dim(X) + \dim(C) = \dim(Y)$.

Theorem

- *If* $m > 6$, *then* $\dim(X) + \dim(C) \neq \dim(Y)$.

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- *If* $m > 6$, *then* $\dim(X) + \dim(C) \neq \dim(Y)$.
- *There are exactly 39 PLPs with* $\dim(X) + \dim(C) = \dim(Y)$:

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$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

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of p points and ℓ lines
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Lemma
A PLP with $\dim(X) + \dim(C) = \dim(Y)$ *is minimal if and only if its joint camera map* $X \times C \to Y$ *is dominant.*

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A map $\varphi : A \to B$ is **surjective** if for every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

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A map $\varphi : A \to B$ is **dominant** if for almost every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

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A map $\varphi : A \to B$ between variety A and B is dominant if and only if for almost every $a \in A$ the differential $D_a\varphi : T_aA \to T_{\varphi(a)}B$ is surjective.

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X	$\times$	C	$\longrightarrow$	Y
(3D-arrangement of p points and ℓ lines with incidences $\mathcal{I}$	,	$\text{cam}_1, \ldots, \text{cam}_m$)	$\longmapsto$	($\text{2D-arr}_1, \ldots, \text{2D-arr}_m$)

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Can check this computationally! It is only linear algebra!

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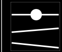
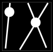
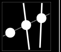
$(p, l, \mathcal{I})$													
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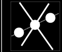
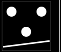
$(p, l, \mathcal{I})$													
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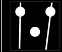
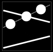
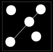
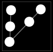
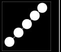
$(p, l, \mathcal{I})$													
Degree	312	224	40	144	144	144	64		20	16	12		

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^f p^d l^f l^a_\alpha$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^f p^d l^f l^a_\alpha$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^f p^d l^f l^a_\alpha$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Degree	312	224	40	144	144	144	64		20	16	12		

- For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}} p^{\mathrm{d}} l^{\mathrm{f}} l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}} p^{\mathrm{d}} l^{\mathrm{f}} l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\mathrm{f}} p^{\mathrm{d}} l^{\mathrm{f}} l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Degree	312	224	40	144	144	144	64		20	16	12		

- For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)
- For $m \in \{4, 5, 6\}$: compute number of solutions with **homotopy continuation** and **monodromy** (state-of-the-art method in numerical algebraic geometry)