Metric Algebraic Geometry Tutorial

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This tutorial is based on Chapters 6, 7 & 8 of the textbook *Metric Algebraic Geometry* by P. Breiding, K. Kohn, B. Sturmfels (Oberwolfach Seminars, Birkhäuser 2024). Many results in those chapters are due to other excellent mathematicians (see the book for references). Also several of the figures in the following slides were created by others (M. Brandt, G. Marchetti, V. Shahverdi, M. Weinstein).

Teaser – Training Neural Networks

A Shallow Neural Network

$$\mu : \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} e & f \end{bmatrix} \sigma\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

- the activation function $\sigma(X) = X^4$ gets applied entrywise
- $a, b, \ldots, f$ are the learnable parameters

This parametrizes quartic homogeneous polynomials in (x, y):

$$Ax^4 + Bx^3y + Cx^2y^2 + Dxy^3 + Ey^4.$$

The Zariski closure of the set of all parametrized polynomials is a 3-fold in $\mathbb{P}^4$:

$$2C^3 - 9BCD + 27AD^2 + 27B^2E - 72ACE = 0.$$

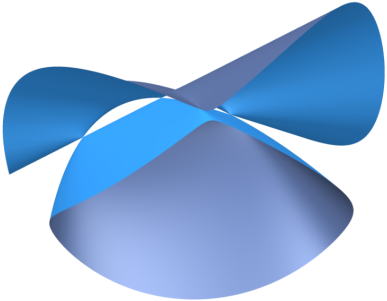


Figure: $C = 1, A + B = D + E$

Neuromanifold & Network Training

$$(a, b, \ldots, f) \longmapsto \mu(x, y) = \begin{bmatrix} e & f \end{bmatrix} \sigma \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \in \mathrm{Sym}_4(\mathbb{R}^2)$$

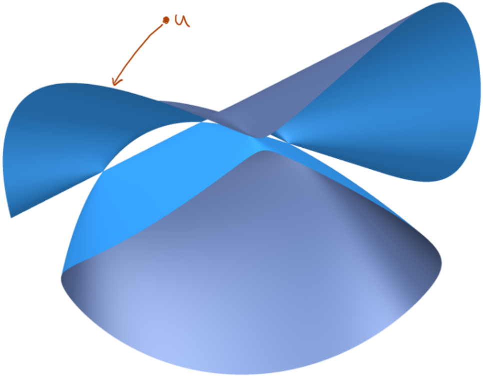
The image of this map is a proper semi-algebraic set, called the **neuromanifold** $\mathcal{M}$ of the network (although it has singularities!)

Let's train the network by minimizing the mean squared error loss for given training data $\mathcal{D} = \{(x_1, y_1, z_1), \ldots, (x_1, y_1, z_d)\}$:

$$\arg\min_{\mu \in \mathcal{M}} \sum_{i=1}^{d} (z_i - \mu(x_i, y_i))^2$$

Proposition:

$$\arg\min_{\mu\in\mathcal{M}} \sum_{i=1}^{d}(z_i - \mu(x_i, y_i))^2 = \arg\min_{\mu\in\mathcal{M}} (\mu - u)^\top Q(\mu - u), \quad \text{where}$$

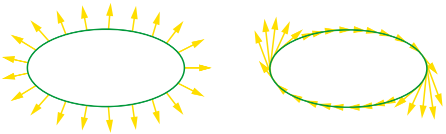


$$Q := V^\top V,\ u := V^+ z,$$

$$V := \begin{bmatrix} x_1^4 & x_1^3 y_1 & x_1^2 y_1^2 & x_1 y_1^3 & y_1^4 \\ x_2^4 & x_2^3 y_2 & x_2^2 y_2^2 & x_2 y_2^3 & y_2^4 \\ & & \vdots & & \\ x_d^4 & x_d^3 y_d & x_d^2 y_d^2 & x_d y_d^3 & y_d^4 \end{bmatrix}$$

Curvature & Volumes of Tubes

Plane Curves & Curvature



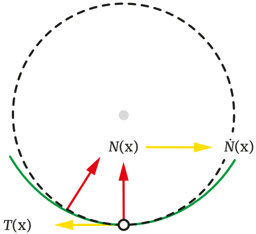
- Let $C = \{f(x_1, x_2) = 0\} \subset \mathbb{R}^2$, $\nabla f(x) \neq 0$ on C.
- Unit normal and tangent fields:

$$N(x) = \frac{\nabla f(x)}{\|\nabla f(x)\|}, \qquad T(x) = (N_2(x), \, -N_1(x)).$$

- Signed curvature

$$c(x) \; = \; \langle T, \, T_1 \, \partial_{x_1} N + T_2 \, \partial_{x_2} N \rangle = \frac{T^T \, H \, T}{\|\nabla f\|},$$

where H is the Hessian of f.



Regions of high curvature are often critical points of distance minimization!

Evolute, Inflections & Critical Curvature

- Radius of curvature $r(x) = 1/c(x)$, center of curvature

$$\Gamma(x) = x - r(x)\,N(x).$$

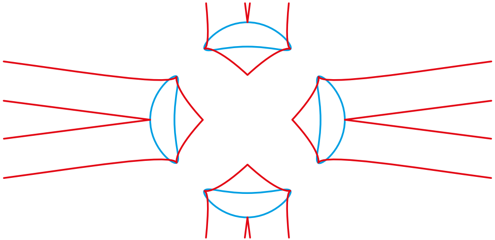
- The *evolute / ED discriminant* E is the Zariski-closure of all centers $\Gamma(x)$.
- Special points on C:

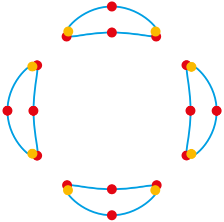
  Inflection point:

$$c(x) = 0 \;\Leftrightarrow\; \Gamma(x) \text{ at infinity.}$$

  Critical curvature:

$$\nabla c(x) \perp T(x) \;\Leftrightarrow\; \text{ cusp on } E.$$

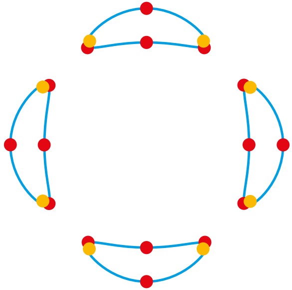




On the ED discriminant, critical points of Euclidean distance collide.

Counting Inflection & Critical Points

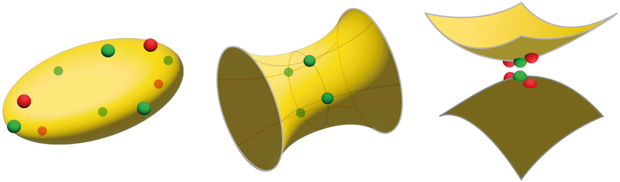
- Homogenize $f \to F(x_0, x_1, x_2)$. Let H_0 be its 3×3 Hessian.
- Curvature formula
$$c(x) = \frac{-\det H_0}{(d-1)^2\,(f_1^2 + f_2^2)^{3/2}}\bigg|_{x_0=1},$$
where $d = \deg f$ and $f_i = \frac{\partial f}{\partial x_i}$.
- Inflection points: $f = \det H_0 = 0$.
$$\text{By Bézout: } \#_{\mathbb{C}} = 3d(d-2),$$
$$\text{By Klein: } \#_{\mathbb{R}} \leq d(d-2).$$
- Critical curvature:
$$\#_{\mathbb{C}} = 2d(3d-5).$$
- Example (Trott curve, $d = 4$): 8 real inflections, 24 real critical points.



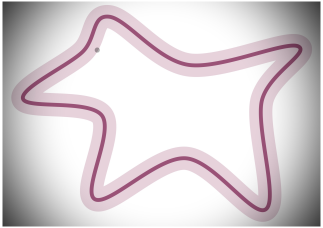
- Let $X \subset \mathbb{R}^n$ be cut out by $f_1, \ldots, f_k$, Jacobian $J = \big(\nabla f_1(x) \cdots \ \nabla f_k(x) \big)$.
- A normal vector $v = J\,w \neq 0$, unit normal $N = v/\|v\|$. Tangent $t \in T_x X$.
- *Curvature in direction* (t, v):

$$c(x, t, v) \;=\; \frac{1}{\|v\|}\, t^T\Big(\sum_{i=1}^{k} w_i\, H_i\Big)\, t.$$

- This quadratic form on $T_x X$ is the *second fundamental form* II_v.
- Its self-adjoint linear map is the *Weingarten map* L_v.
  Eigenvalues $=$ principal curvatures.



Volumes of Tubular Neighborhoods



Tube of radius ε:

$$\mathrm{Tube}(X,\varepsilon) = \{u \in \mathbb{R}^n \mid \min_{x\in X} \|x-u\| < \varepsilon\}.$$

For X a neuromanifold, the volume of the tube measures the expressivity of the neural network!

Let $X \subset \mathbb{R}^n$ be smooth and compact.

- The *reach* of X is the supremum over all $\varepsilon > 0$ such that the exponential map

  $$\varphi_\varepsilon : \mathcal{N}_\varepsilon X = \{(x,v) \mid x \in X, v \perp T_x X, \|v\| < \varepsilon\} \to \mathrm{Tube}(X,\varepsilon),\ (x,v) \mapsto x+v$$

  is a diffeomorphism.
- For $\varepsilon <$ the reach of X: *Weyl's tube formula:*

  $$\mathrm{vol}\big(\mathrm{Tube}(X,\varepsilon)\big) = \sum_{0\le 2i\le m} \kappa_{2i}(X)\,\varepsilon^{\,n-m+2i}, \quad m = \dim(X),$$

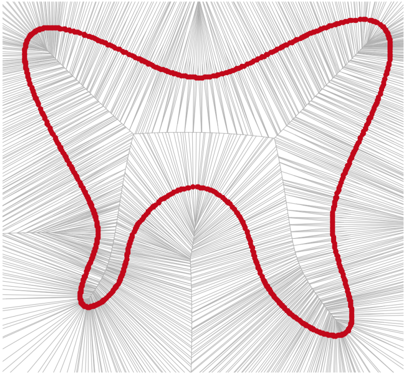
  where κ_{2i} are integrals of the $2i$-minors of the Weingarten map L_w.

Medial Axis & Offset

Medial Axis

The *medial axis* $\mathrm{Med}(X) \subset \mathbb{R}^n$ is the set of points having *at least two* distinct closest points on X.

If X is semialgebraic then so is $\mathrm{Med}(X)$.



Proposition:

$$\mathrm{dist}\big(X, \mathrm{Med}(X)\big) \;=\; \mathrm{reach}(X).$$

Hence points within distance $< \mathrm{reach}(X)$ from X have a unique nearest point on X.

Bottlenecks, Curvature, and Reach

- A *bottleneck* is a pair $\{x, y\} \subset X$, $x \neq y$, for which $x - y$ is normal to both T_xX and T_yX.
- Its *width* is $b(x, y) = \frac{1}{2}\|x - y\|$.

$$B(X) = \min_{\text{bottlenecks}} b(x, y).$$

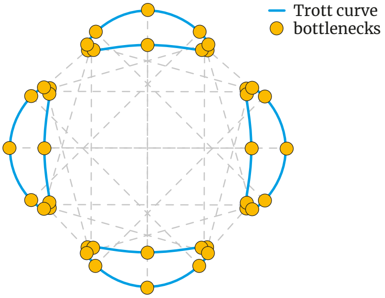
- The *maximal curvature* of X is

$$C(X) = \max_{x \in X} \max_i |c_i(x)|,$$

where $c_i(x)$ are principal curvatures at x.

Theorem: For X smooth,

$$\mathrm{reach}(X) \ = \ \min\{B(X), \, 1/C(X)\}.$$



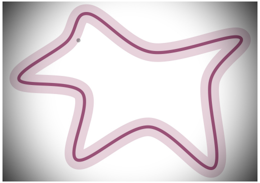
Offset Hypersurfaces & Offset Polynomial

- Let $X \subset \mathbb{R}^n$ be irreducible. Its *ED correspondence* is

$$\mathcal{E}_X = \overline{\{(x,u) \mid x \in X,\ u - x \perp T_x X\}} \subset X \times \mathbb{C}^n.$$

- *Offset correspondence*:

$$\mathcal{OC}_X = \{(x,u,\varepsilon) \in \mathcal{E}_X \times \mathbb{C} \mid \|u-x\|^2 = \varepsilon^2\}.$$



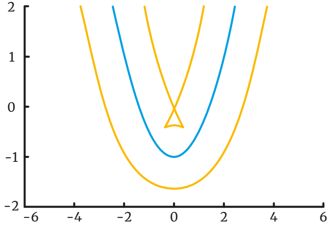
- The closure of its projection to (u,ε) is the *offset hypersurface*

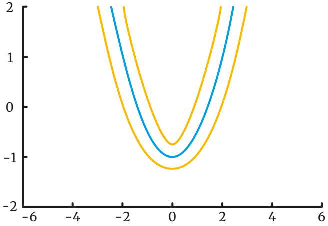
$$\mathrm{Off}_X \subset \mathbb{C}^n \times \mathbb{C}, \quad \mathrm{codim} = 1.$$

- Hence there is a defining *offset polynomial*

$$g_X(u,\varepsilon) = 0.$$

Offset Hypersurface of the Parabola





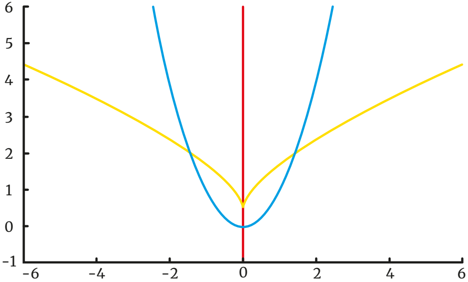
Offset Discriminant & its Decomposition

Define the *offset discriminant* $\delta_X(u) = \mathrm{Disc}_\varepsilon\big(g_X(u,\varepsilon)\big)$ and

$$\Delta_X^{\mathrm{Off}} = V(\delta_X) \subset \mathbb{C}^n.$$

- A point u lies in Δ_X^{Off} iff
  - it has a multiple critical value ($u \in \Sigma_X$, the ED discriminant),
  - or two distinct critical points lie at equal distance (the bisector hypersurface Bis_X).
- Theorem (Horobeț–Weinstein): Write $M_X := \overline{\mathrm{Med}(X)}$. Then

$$\Delta_X^{\mathrm{Off}} = \mathrm{Bis}_X \cup \Sigma_X \ \supseteq \ X \cup M_X \cup \Sigma_X.$$



Computing Normals & Curvature from the Offset Polynomial

- For $u \notin \Delta_X^{\mathrm{Off}}$, let $\varepsilon(u)$ be a local real root of $g_X(u, \varepsilon) = 0$. Suppose that $x \in X$ is the critical point corresponding to (u, ε). By implicit differentiation,

$$\nabla_u \, \varepsilon(u) = -\left(\tfrac{\partial g_X}{\partial \varepsilon}\right)^{-1} \tfrac{\partial g_X}{\partial u},$$

  which is a unit normal vector at x on X.

- Differentiating $\nabla_u \, \varepsilon(u)$ in direction $t \in T_x X$ gives the second fundamental form evaluated at t. This means:

$$\mathrm{II}_{u-x}(t) \;=\; \lim_{\substack{s \to 0 \\ s > 0}} \; t^\top \left(\frac{\partial^2 \varepsilon}{\partial u^2}(x + s(u - x), \; s\varepsilon) \right) t.$$

- **Conclusion:** from g_X one extracts both the normal field and all principal curvatures of X.

example: parabola

For $X = V(x_2 - x_1^2)$, we find $\frac{d\varepsilon}{du}(u,\varepsilon) = \frac{1}{p}(h_1, h_2)$, where

$$h_1 = -96u_1\varepsilon^4 + \left(192u_1^3 + 64u_1u_2^2 - 16u_1u_2 + 40u_1\right)\varepsilon^2 - 4u_1\left(u_1^2 - u_2\right)\left(24u_1^2 + 16u_2^2 - 16u_2 + 1\right)$$

$$\begin{aligned} h_2 = {} & (-32u_2 - 32)\,\varepsilon^4 + \left(64u_1^2u_2 - 8u_1^2 + 96u_2^2 + 16u_2 - 8\right)\varepsilon^2 \\ & - 2\left(u_1^2 - u_2\right)\left(16u_1^2u_22 - 20u_1^2 - 32u_2^2 + 12u_2 - 1\right) \end{aligned}$$

$$\begin{aligned} p = {} & -96\varepsilon^5 + \left(192u_1^2 + 64u_2^2 + 128u_2 - 32\right)\varepsilon^3 \\ & + \left(-96u_1^4 - 64u_1^2u_2^2 + 16u_1^2u_2 - 64u_2^3 - 40u_1^2 - 16u_2^2 + 16u_2 - 2\right)\varepsilon. \end{aligned}$$

example:

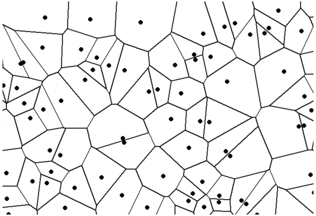
- for $u = (0, \frac{1}{4})$ and $\varepsilon = \frac{1}{4}$, this computes the unit normal $(0,1)$ at $x = (0,0)$
- the Hessian matrix of $\varepsilon(u)$ is a large expression
- evaluated at $(su, s\varepsilon) = (0, \frac{s}{4}, \frac{s}{4})$ and letting $s \to 0$ yields $A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$

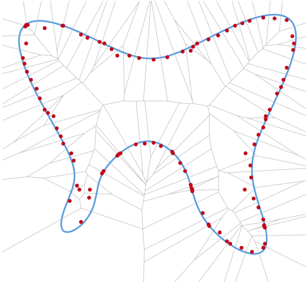
Voronoi Cells

Voronoi Cells

Definition: Let $X \subset \mathbb{R}^n$ and fix $y \in X$. The *Voronoi cell* of y is

$$\mathrm{Vor}_X(y) \;=\; \{u \in \mathbb{R}^n \mid y \in \arg\min_{x \in X} \|u - x\|\}.$$



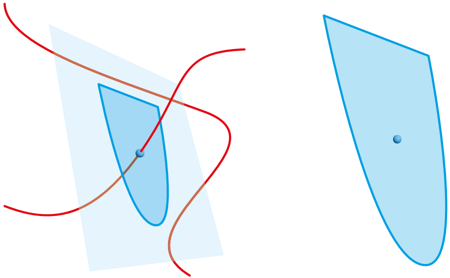


The union of the boundaries of the Voronoi cells is the medial axis.

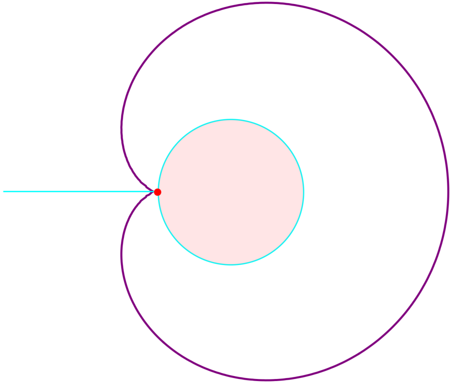
Proposition: $X \subset \mathbb{R}^n$ algebraic variety, $y \in X$ is smooth. Then $\mathrm{Vor}_X(y)$ is a full-dimensional, convex, semialgebraic subset of the *affine normal space*

$$\begin{aligned} N_X(y) &= y + N_y X \\ &= \{u \mid u - y \perp T_y X\}. \end{aligned}$$

Voronoi Cells & Singularities



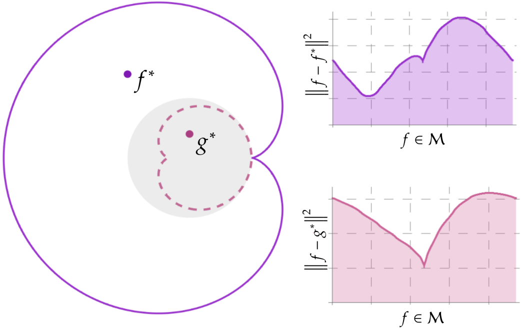
at a smooth point of a space curve



the Voronoi cell at the singularity is 2-dimensional, i.e., that point is the closest with **positive** probability! (medial axis)

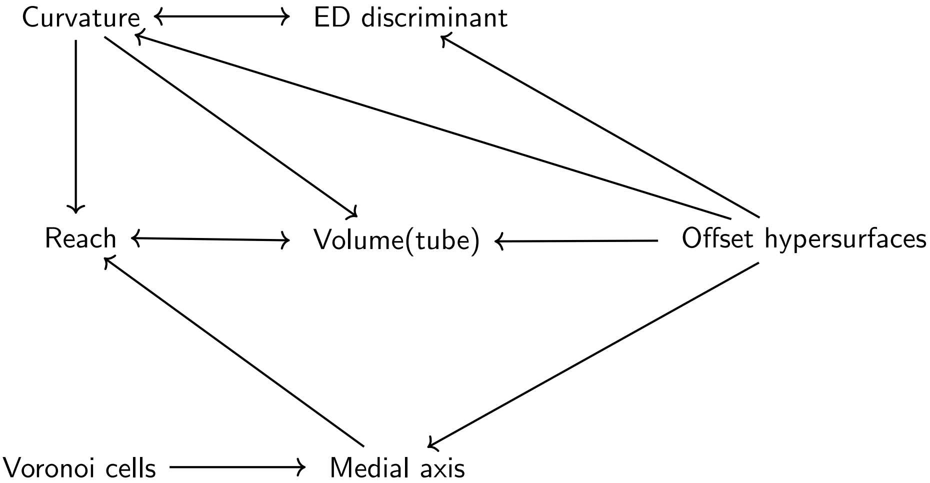
Singularities of neuromanifolds can cause implicit biases.

Voronoi Cells & ED discriminant



The number or type of critical points change when crossing the medial axis or the **ED discriminant**.

An Overview


Curvature
ED discriminant
Reach
Volume(tube)
Offset hypersurfaces
Voronoi cells
Medial axis

Open Problems

Many theoretical and computational questions remain!
Check out our accompanying exercises. :)