POLAR DEGREES

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The goal of this lecture is to provide three definitions of *polar degrees*, in terms of non-transversal intersections, Schubert varieties and the Gauss map, and conormal varieties. We recall the biduality theorem of projective varieties and discuss the key properties of polar degrees under projective duality. Finally, we explain how polar degrees are related with Chern classes.

We work over an algebraically closed field of characteristic zero.

1. Polar varieties

Example 1. Imagine that you look at an algebraic surface $X \subseteq \mathbb{P}^3$ from a point $V \in \mathbb{P}^3$. If you would want to sketch the surface from your point of view, you would draw its *contour curve* $P(X, V)$; see Figure 1. The contour curve consists of all points p on the surface X such that the line spanned by V and p is tangent at p. The *first polar degree* $\mu_1(X)$ of the surface X is the degree of the contour curve for a generic point V.

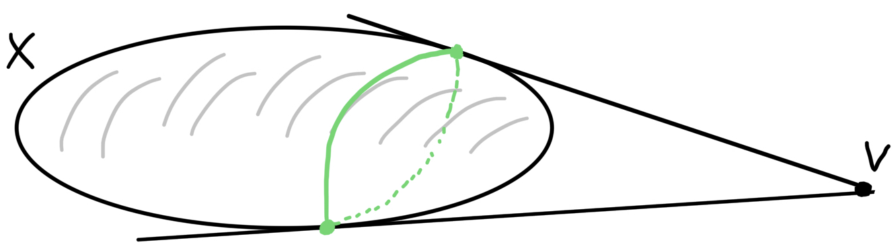

Figure 1. Green contour curve when the ellipsoid X is viewed from the point V.

Now we change the setting slightly and imaging that our viewing of the surface X is not centered at a point but at a line $V \subseteq \mathbb{P}^3$. This time our contour set $P(X, V)$ consists of all point p on the surface X such that the plane spanned by the point p and the line V is tangent at p; see Figure 2. For a generic line V, the contour set $P(X, V)$ is finite, and the *second polar degree* is its cardinality. ♢

The contour sets described in the example above are also known as *polar varieties*. To define polar varieties in general, we need to fix some conventions and notations. For instance, the dimension of the empty set is considered to be -1. Given projective subspaces $V, W \subseteq \mathbb{P}^n$, their projective span (equivalently, their join) is denoted by $V + W \subseteq \mathbb{P}^n$.

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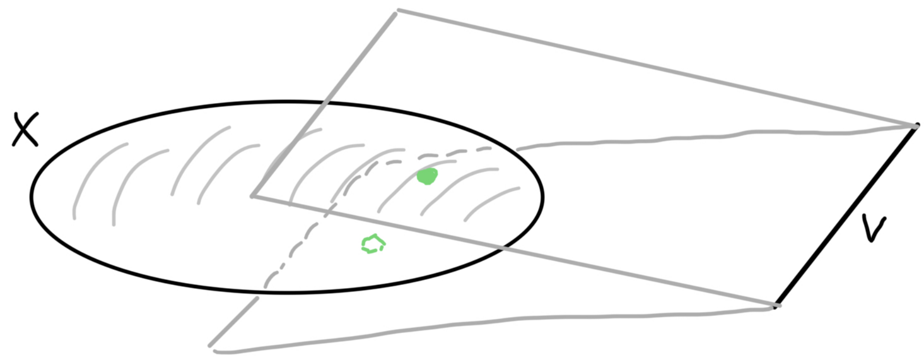

FIGURE 2. The green contour set consists of two points when the ellipsoid X is viewed from the line V.

Given a projective variety $X \subseteq \mathbb{P}^n$, we write $\mathrm{Reg}(X)$ for its regular locus. The *embedded tangent space* of X at $p \in \mathrm{Reg}(X)$ is

$$\mathbb{T}_p X := \left\{ v \in \mathbb{P}^n \mid \forall f \in I(X) : \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(p) \cdot v_i = 0 \right\}.$$

We recall that a projective subspace $W \subseteq \mathbb{P}^m$ is said to intersect X *non-transversely* at $p \in \mathrm{Reg}(X)$ if $p \in W$ and $\dim(W + \mathbb{T}_p X) < n$.

Definition 2. Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety. The *polar variety* of X with respect to a projective subspace $V \subseteq \mathbb{P}^n$ is

$$P(X, V) := \overline{\{p \in \mathrm{Reg}(X) \setminus V \mid V + p \text{ intersects } X \text{ at } p \text{ non-transversely}\}}.$$

For every $i \in \{0, \ldots, \dim X\}$, there is an integer $\mu_i(X)$ that is equal to the degree of $P(X, V)$ for almost all projective subspaces $V \subseteq \mathbb{P}^n$ with $\dim V = \operatorname{codim} X - 2 + i$. $\mu_i(X)$ is called the *i-th polar degree of X*.

Example 3. A surface $X \subseteq \mathbb{P}^3$ has three polar degrees. For $i = 0, 1, 2$, the generic subspace V in Definition 2 is empty, a point, or a line, respectively. We have seen the latter two cases in Example 1. For the case $i = 0$, we observe that $P(X, \emptyset) = X$, and so the 0-th polar degree $\mu_0(X)$ is the degree of the surface X. $\diamondsuit$

Example 4. The last observation that $\mu_0(X) = \deg(X)$ is true in general. If $i = 0$, the dimension of the generic subspace V in Definition 2 is $\operatorname{codim} X - 2$. Hence, we have for every $p \in \mathrm{Reg}(X)$ that

$$\begin{aligned} \dim\left((V + p) + \mathbb{T}_p X\right) &= \dim\left(V + \mathbb{T}_p X\right) = \dim V + \dim X - \dim(V \cap \mathbb{T}_p X) \\ &\leq (\operatorname{codim} X - 2) + \dim X + 1 = n - 1, \end{aligned}$$

which means that $V + p$ intersects X at p non-transversely. Therefore, we conclude that $P(X, V) = X$ and $\mu_0(X) = \deg(X)$. $\diamondsuit$

We will now give a second definition of polar varieties in terms of the Gauss map and Schubert varieties. For that fix a projective subspace $V \subseteq \mathbb{P}^n$. We observe that

$V + p$ intersects X at $p \in \mathrm{Reg}(X)$ non-transversely (i.e., $n > \dim((V+p) + \mathbb{T}_pX) = \dim V + \dim X - \dim(V \cap \mathbb{T}_pX))$ if and only

$$\dim(V \cap \mathbb{T}_pX) > \dim V - \operatorname{codim} X. \tag{1}$$

Since $\dim V - \operatorname{codim} X$ is the expected dimension of the intersection of the two projective subspaces V and $\mathbb{T}_pX$, condition (1) means that the tangent space $\mathbb{T}_pV$ meets V in an unexpectedly large dimension. Such subspaces are collected in simple instances of *Schubert varieties*:

$$\Sigma_m(V) := \{T \in \mathrm{Gr}(m, \mathbb{P}^n) \mid \dim(V \cap T) > \dim V - n + m\}.$$

If $m = \dim X$, then condition (1) is equivalent to $\mathbb{T}_pX \in \Sigma_m(V)$. Hence, the *Gauss map*

$$\begin{aligned} \gamma_X : X &\dashrightarrow \mathrm{Gr}(m, \mathbb{P}^n), \\ p &\longmapsto \mathbb{T}_pX \end{aligned}$$

pulls the Schubert variety $\Sigma_m(V)$ back to the polar variety $P(X, V)$, i.e.,

$$P(X, V) = \overline{\gamma_X^{-1}\left(\Sigma_{\dim X}(V)\right)}.$$

2. Projective Duality

We recall that there is a one-to-one correspondence between hyperplanes H in the $\mathbb{P}^n$ and points $H^\vee$ in the dual projective space $(\mathbb{P}^n)^*$. The *dual variety* of a projective variety $X \subseteq \mathbb{P}^n$ consists of all tangent hyperplanes of X:

$$X^\vee := \overline{\{H^\vee \in (\mathbb{P}^n)^* \mid \exists p \in \mathrm{Reg}(X) : \mathbb{T}_pX \subseteq H\}}.$$

Theorem 5 (Biduality theorem, [GKZ94])**.** *Let $X \subseteq \mathbb{P}^n$ be a projective variety over an algebraically closed field of characteristic zero. Moreover, let $p \in \mathrm{Reg}(X)$ and $H^\vee \in \mathrm{Reg}(X^\vee)$. The hyperplane H is tangent to X at the point p if and only if the hyperplane $p^\vee$ is tangent to $X^\vee$ at the point $H^\vee$. In partcular, $(X^\vee)^\vee = X$.*

Example 6. Let us revisit the example in Figure 2 that illustrates the polar variety $P(X, V)$ of a surface $X \subseteq \mathbb{P}^3$ and a generic line V. The tangent planes passing through the line V correspond in $(\mathbb{P}^3)^*$ to points on the dual variety $X^\vee$ that are contained in the line $V^\vee$; see Figure 3. Hence, if the dual variety is a surface as well, then its degree is given by the second polar degree of X, i.e., $\mu_2(X) = \deg(X^\vee)$. Otherwise, if the dual variety $X^\vee$ is of smaller dimension, the line $V^\vee$ misses it and $\mu_2(X) = 0$.

In the setting of Figure 1, the polar variety $P(X, V)$ of a surface $X \subseteq \mathbb{P}^3$ and a generic point V is a curve. Thus, its degree ($= \mu_1(X)$) is computed by intersecting it with a generic plane H. The polar curve consists of all points on X whose tangent plane contains the point V, i.e., $P(X, V) = \overline{\{p \in \mathrm{Reg}(X) \mid V \in \mathbb{T}_pX\}}$. Hence, the first polar degree $\mu_1(X)$ counts all (regular) points $p \in X$ such that

$$p \in H \text{ and } V \in \mathbb{T}_pX. \tag{2}$$

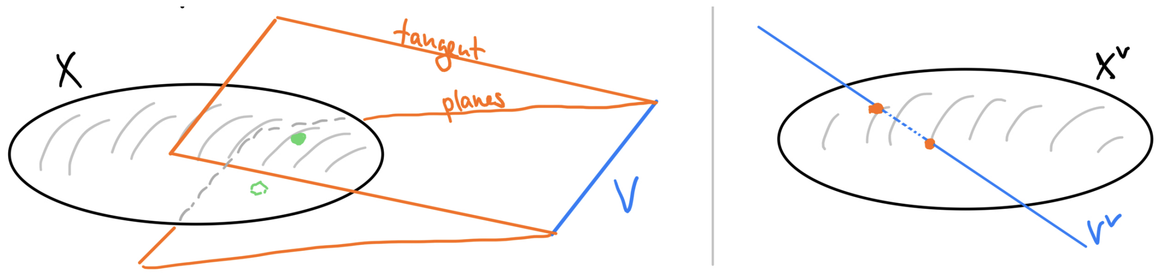

FIGURE 3. $\mu_2(X) = \deg(X^\vee)$ holds for pairs of dual surfaces in projective 3-space ($q_i = (\mathbb{T}_{p_i}X)^\vee$).

The tangent planes at those points correspond to points $q := (\mathbb{T}_pX)^\vee$ in the dual projective space. By the biduality theorem, those points satisfy $\mathbb{T}_qX^\vee = p^\vee$ if the dual variety $X^\vee$ is a surface. Hence, in that case, the two conditions in (2) are equivalent to

$$H^\vee \in \mathbb{T}_qX^\vee \text{ and } q \in V^\vee. \tag{3}$$

Comparing now (3) with (2), we see that the point-plane pair $(H^\vee, V^\vee)$ imposes the same conditions on the points $q \in X^\vee$ as the point-plane pair (V, H) imposes on the points $p \in X$; see also Figure 4. Due to the genericity of (V, H), we conclude that $\mu_1(X) = \mu_1(X^\vee)$ if $X^\vee$ is a surface. If $X^\vee$ is a curve, then $\mathbb{T}_qX^\vee \subsetneq p^\vee$ and so the only conditions imposed by (2) on the points $q \in X^\vee$ is that they must lie in $V^\vee$, i.e., $\mu_1(X) = |X^\vee \cap V^\vee| = \deg(X^\vee) = \mu_0(X^\vee)$. Finally, if $X^\vee$ is a point (i.e., X is a plane), then $\mu_1(X) = 0$. ♢

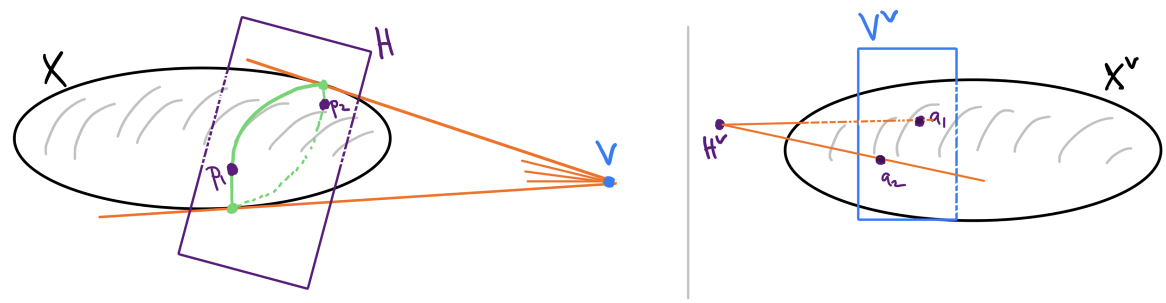

FIGURE 4. $\mu_1(X) = \mu_1(X^\vee)$ holds for pairs of dual surfaces in projective 3-space.

The relations between the polar degrees of a variety and its dual that we observed in the previous example are true in more generality. More specifically, the list of polar degrees of a projective variety X is exactly the list of polar degrees of its dual variety $X^\vee$ in reversed order. As seen in Example 4, the first non-zero entry in that list is $\deg(X)$, and so its last non-zero entry is $\deg(X^\vee)$. We summarize these key properties of polar degrees:

Proposition 7 ([Hol88])**.** *Let X be an irreducible projective variety, and let $\alpha(X) := \dim X - \operatorname{codim} X^\vee + 1$.*

(a) $\mu_i(X) > 0 \quad \Leftrightarrow \quad 0 \leq i \leq \alpha(X)$.
(b) $\mu_0(X) = \deg X$.
(c) $\mu_{\alpha(X)}(X) = \deg X^\vee$.
(d) $\mu_i(X) = \mu_{\alpha(X)-i}(X^\vee)$.

The ideas discussed in Example 6 can be turned into formal proofs for almost all assertions in Proposition 7 (only the direction "$\Leftarrow$" in (a) is rather tricky). Another strategy is to first establish the relation of the polar degrees with the *conormal variety* of the projective variety $X \subseteq \mathbb{P}^n$:

$$\mathcal{N}_X := \overline{\{(p, H^\vee) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \mid p \in \mathrm{Reg}(X), \mathbb{T}_p X \subseteq H\}}.$$

The projection of the conormal variety $\mathcal{N}_X$ onto the first resp. second factor is the variety X resp. its dual $X^\vee$. Moreover, by the biduality theorem, we have that

$$\mathcal{N}_{X^\vee} = \{(H^\vee, p) \mid (p, H^\vee) \in \mathcal{N}_X\}. \tag{4}$$

Independently of the dimension of X, the dimension of the conormal variety is always $n-1$. Hence, the *multidegree* of the conornomal variety is given by its intersections with $L_1 \times L_2$, where $L_1 \subseteq \mathbb{P}^n$ and $L_2 \subseteq (\mathbb{P}^n)^*$ are generic subspaces with $\dim L_1 + \dim L_2 = \operatorname{codim} \mathcal{N}_X = n+1$. We denote the entries of that multidegree by

$$\delta_j(X) := |\mathcal{N}_X \cap (L_1 \times L_2)|, \text{ for generic } L_1, L_2 \text{ with } \dim L_2 = j, \dim L_1 = n+1-j.$$

They are in fact the polar degrees:

Proposition 8 ([Kle86, Prop. (3) on page 187] or [FKM83, Lem. (2.23) on page 169])**.** $\delta_j(X) = \mu_i(X)$, *where* $i := \dim X + 1 - j$.

This proposition together with (4) implies immediately Proposition 7(d) and hence using Example 4 also (b) and (c). The direction "$\Rightarrow$" in Proposition 7(a) can also be deduced directly from the definition of the $\delta_j(X)$.

Before we present an idea of proof for Proposition 7, we revisit our running example.

Example 9. We see from Figure 4 and the conditions (2) and (3) that the first polar degree of a surface X in $\mathbb{P}^3$ is computed as $\mu_1(X) = |\mathcal{N}_X \cap (H \times V^\vee)| = \delta_2(X)$. In other words, Proposition 7 holds for surfaces X in $\mathbb{P}^3$.

Proof idea for Proposition 7. Let $L_1 \subseteq \mathbb{P}^n$ and $L_2 \subseteq (\mathbb{P}^n)^*$ be generic subspace of dimensions $n+1-j$ and j, respectively. Setting $V := L_2^\vee$, we start by observing that V has the correct dimension to be used in the computation of the i-th polar degree (where $i = \dim X + 1 - j$), since $\dim V = n - j - 1 = \operatorname{codim} X - 2 + i$.

Now we consider a generic pair $(p, H^\vee) \in \mathcal{N}_X \cap (\mathbb{P}^n \times L_2)$. The point $p \in X$ is regular and both its tangent space $\mathbb{T}_p X$ and $V = L_2^\vee$ are contained in the hyperplane H. In particular, we have $\dim(V + \mathbb{T}_p X) < n$ and so p is in the polar variety $P(X, V)$. In fact, the projection $\mathcal{N}_X \cap (\mathbb{P}^n \times L_2) \to P(X, V)$ onto the first factor is birational. Hence, $\mu_i(X) = \deg P(X, V) = |P(X, V) \cap L_1| = |\mathcal{N}_X \cap (L_1 \times L_2)| = \delta_j(X)$. $\square$

3. Chern Classes

Chern classes are topological invariants associated with vector bundles on smooth manifolds or varieties. For a smooth, irreducible projective variety X, its polar degrees can be computed from its Chern classes (see Proposition 10).

To a vector bundle $\mathcal{E}$ on X of rank r, we associate the Chern classes $c_0(\mathcal{E}), \ldots, c_r(\mathcal{E})$, which are formally elements in the *Chow ring* of X. Chern classes are easiest understood when the vector bundle $\mathcal{E}$ is globally generated. In that case, the Chern class $c_{r+1-j}(\mathcal{E})$ is the element in the Chow ring of X that is associated with the following degeneracy locus:

$$D(\sigma_1, \ldots, \sigma_j) := \{x \in X \mid \sigma_1(x), \ldots, \sigma_j(x) \text{ are linearly dependent}\},$$

where $\sigma_1, \ldots, \sigma_j : X \to \mathcal{E}$ are j general global sections. For the purpose of this section, it is not crucial to understand the Chow ring. It suffices for us to understand the *degree* of $c_{r+1-j}(\mathcal{E})$, which is the degree of the degeneracy locus $D(\sigma_1, \ldots, \sigma_j)$ for general σ_i. For instance, the degree of the *top Chern class* $c_r(\mathcal{E})$ is the degree of the vanishing locus of a single general global section.

There are some calculation rules that allow us to compute Chern classes of more complex vector bundles. Most notably, the *Whitney sum formula* states for a short exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of vector bundles that $c_k(\mathcal{E}) = \sum_{i+j=k} c_i(\mathcal{E}')c_j(\mathcal{E}'')$; see [Ful98, Theorem 3.2]. The Chern class $c_k(X)$ of X is an abbreviation for the Chern class $c_k(\mathcal{T}X)$ of its tangent bundle $\mathcal{T}X$.

Proposition 10 ([Hol88, eq. (3)])**.** *Let X be a smooth, irreducible projective variety, and let $m := \dim X$. Then,*

$$\mu_i(X) = \sum_{k=0}^{i} (-1)^k \binom{m-k+1}{m-i+1} \deg(c_k(X)).$$

This formula can also be reverted to express degrees of Chern classes in terms of polar degrees:

$$\deg(c_k(X)) = \sum_{i=0}^{k} (-1)^k \binom{m-i+1}{m-k+1} \mu_i(X). \tag{5}$$

Remark 11. Both formulas also hold for singular varieties, after replacing the classical Chern classes with Chern-Mather classes. That result is due to R. Piene (see [Pie88, Theorem 3] or [Pie78]).

An important difference between polar degrees and Chern classes is the following: Polar degrees are projective invariants of the embedded variety $X \subseteq \mathbb{P}^n$. This holds also more generally for the *polar classes*, i.e., the rational equivalence classes (in the Chow ring of X) of the polar varieties. Chern classes are even *intrinsic invariants* of the variety X, i.e., they do not depend on the embedding of X in projective space.

Example 12. Let X be a smooth, irreducible projective variety.

a) We see from (5) that $\deg(c_0(X)) = \mu_0(X) = \deg X$.

b) The top Chern class of X coincides with its topological Euler characteristic: $\deg(c_m(X)) = \chi(X)$, where $m = \dim X$.

c) If X is a curve, then $\chi(X) = 2 - 2g(X)$, where $g(X)$ denotes the genus. Moreover, we see from (5) that $\deg(c_1(X)) = 2\deg X - \mu_1(X)$. Hence, we conclude:

$$2\deg X - \mu_1(X) = 2 - 2g(X). \tag{6}$$

d) If $X \subseteq \mathbb{P}^n$ is a rational curve, we can easily verify the relation (6):
- If X is a line, its dual variety is never a hypersurface, and so $\mu_1(X) = 0$.
- If X is a conic (i.e., $\deg X = 2$), its dual variety is (a cone over) a conic, and so $\mu_1(X) = \deg X^\vee = 2$.
- If X is a twisted cubic (i.e., $\deg X = 3$), its dual variety is (a cone over) the discriminant hypersurface of a cubic polynomial, and so $\mu_1(X) = \deg X^\vee$ is the degree of that discriminant, which is 4.
- More generally, if X is a rational normal curve of degree d, its dual variety is (a cone over) the discriminant hypersurface of a degree-d polynomial, and so $\mu_1(X) = \deg X^\vee$ is the degree of that discriminant, which is $2d - 2$.

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