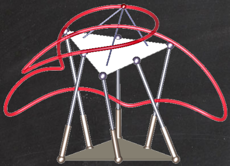
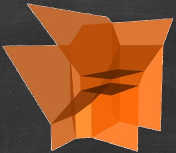
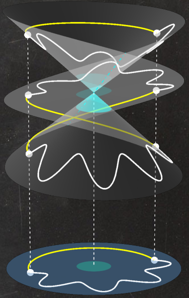


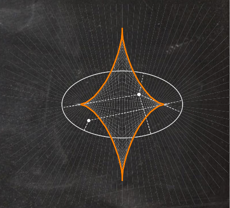




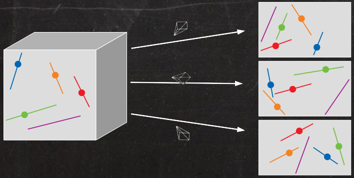
What is Nonlinear Algebra?

Kathlén Kohn

KTH Stockholm



June 10, 2020



Linear algebra

All undergraduate students learn about Gaussian elimination, a general method for solving linear systems of algebraic equations:

Input:

$$\begin{aligned} x + 2y + 3z &= 5 \\ 7x + 11y + 13z &= 17 \\ 19x + 23y + 29z &= 31 \end{aligned}$$

Output:

$$\begin{aligned} x &= -35/18 \\ y &= 2/9 \\ z &= 13/6 \end{aligned}$$

Solving very large linear systems is central to applied mathematics.

Non-linear algebra

Lucky students also learn about Gröbner bases, a general method for non-linear systems of algebraic equations:

Input:

$$\begin{aligned} x^2 + y^2 + z^2 &= 2 \\ x^3 + y^3 + z^3 &= 3 \\ x^4 + y^4 + z^4 &= 4 \end{aligned}$$

Non-linear algebra

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Input:

$$x^2+y^2+z^2=2$$
$$x^3+y^3+z^3=3$$
$$x^4+y^4+z^4=4$$

Output: $3z^{12}-12z^{10}-12z^9+12z^8+72z^7-66z^6-12z^4+12z^3-1=0$

$$\begin{aligned}4y^2+(36z^{11}+54z^{10}-69z^9-252z^8-216z^7+573z^6+72z^5\\ -12z^4-99z^3+10z+3)\cdot y+36z^{11}+48z^{10}-72z^9\\ -234z^8-192z^7+564z^6-48z^5+96z^4-96z^3+10z^2+8=0\end{aligned}$$

$$\begin{aligned}4x+4y+36z^{11}+54z^{10}-69z^9-252z^8-216z^7\\ +573z^6+72z^5-12z^4-99z^3+10z+3=0\end{aligned}$$

This is very hard for large systems, but ...

The world is non-linear!

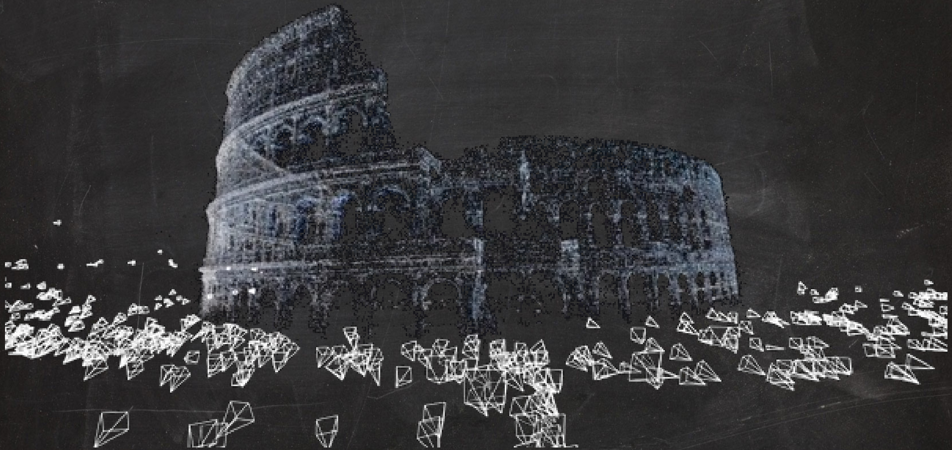
Many models in the sciences and engineering are characterized by polynomial equations. Such a set is an algebraic variety $X \subset \mathbb{R}^n$.

- computer vision
- algebraic statistics
- machine learning
- optimization
- ...

Computer Vision

Structure from Motion

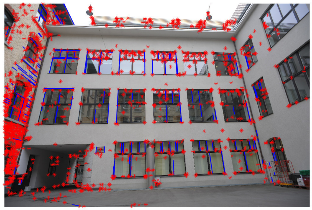
Reconstruct 3D scenes and camera poses from 2D images

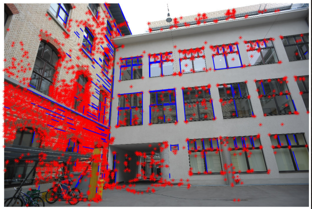


Rome in a Day: S. Agarwal, Y. Furukawa, N. Snavely, I. Simon, S. Seitz, R. Szeliski

Reconstruct 3D scenes and camera poses from 2D images

- Step 1: Identify common points and lines on given images





- Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

Reconstruct 3D scenes and camera poses from 2D images

- Step 1: Identify common points and lines on given images

- Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

$\Rightarrow$ This is an algebraic problem!

What is a camera?

A **camera** is a 3×4 matrix C which takes pictures of points in projective 3-space via

$$\begin{aligned} \mathbb{P}^3 &\longrightarrow \mathbb{P}^2, \\ P &\longmapsto CP. \end{aligned}$$

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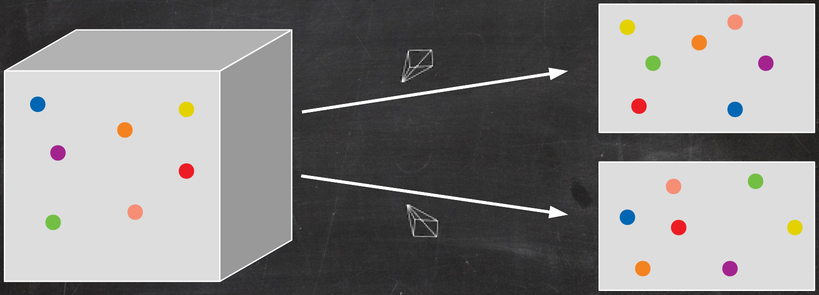
- Each camera matrix C is a point in $\mathbb{P}^{11}$.
- There can be restrictions on the camera matrix C, e.g. by assuming that the focal length of the camera is known.

7-point problem

Given 2 images of 7 points,
can we recover the 7 points in 3D and the 2 cameras?



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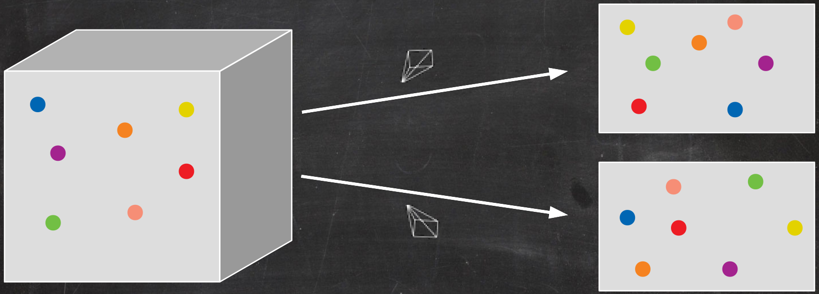

Formally, we study the **joint camera map**

$$\Phi : \left(\mathbb{P}^3\right)^7 \times \left(\mathbb{P}^{11}\right)^2 \dashrightarrow \left(\mathbb{P}^2\right)^{14},$$
$$(P_1, \ldots, P_7, C_1, C_2) \longmapsto (C_1 P_1, \ldots, C_1 P_7, C_2 P_1, \ldots, C_2 P_7),$$

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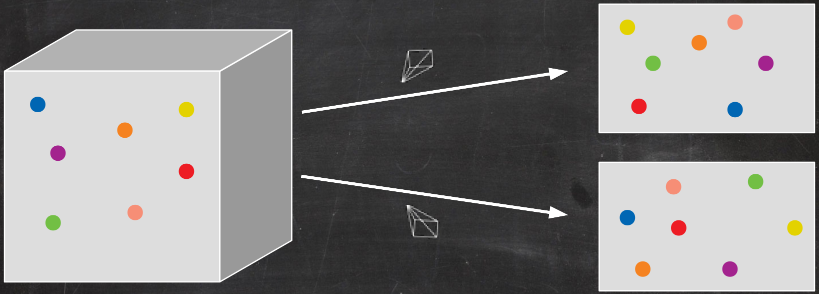

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and given a point in its image $x \in \left(\mathbb{P}^2\right)^{14}$ we ask for its **fiber** $\Phi^{-1}(x)$.

7-point problem

The projective linear group $\mathrm{PGL}(4)$ acts on the fibers $\Phi^{-1}(x)$ via

$$g.(P_1,\ldots,P_7,C_1,C_2) = \left(gP_1,\ldots,gP_7,C_1g^{-1},C_2g^{-1}\right).$$

Practically, this means that we can only hope to recover points and cameras **up to projective transformations**.

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- its fibers are **generically finite**!

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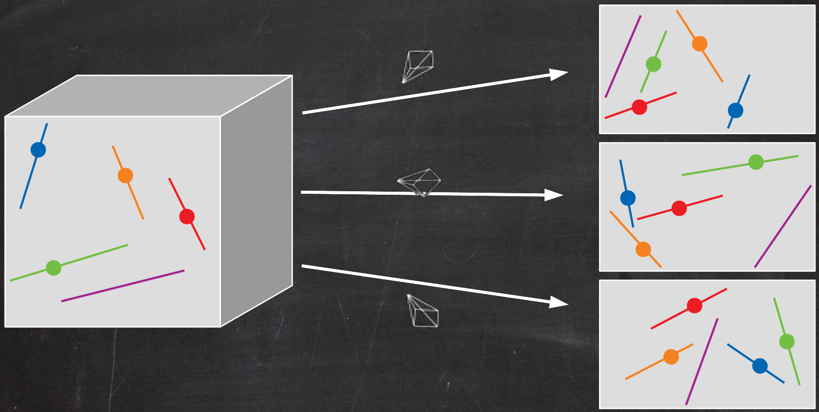
- its fibers are **generically finite**!
  in fact, over $\mathbb{C}$, there are generically **3** solutions to the 7-point problem
- solving naively: **28** quadratic equations in **28** unknowns

A more complicated finite problem



Incidences are modeled by **flag varieties:** $\mathcal{F}_k := \{(P, L) \in \mathbb{P}^k \times \mathrm{Gr}(1, \mathbb{P}^k) \mid P \in L\}$

joint camera map:

$$\Phi : \left(\mathrm{Gr}(1, \mathbb{P}^3) \times (\mathcal{F}_3)^4 \times \left(\mathbb{P}^{11}\right)^3\right) / \mathrm{PGL}(4) \dashrightarrow \mathrm{Gr}(1, \mathbb{P}^2)^3 \times (\mathcal{F}_2)^{12}$$

Algebraic Statistics

Reconstruct probability distributions from moments

Central question:
Let $\{\mu_\theta \mid \theta \in \Theta\}$ be a family of probability distributions on $\mathbb{R}^d$. Can we recover a distribution in the family if we know enough of its **moments**?

$$m_{i_1 i_2 \dots i_d}(\mu_\theta) = \int_{\mathbb{R}^d} w_1^{i_1} w_2^{i_2} \cdots w_d^{i_d} \, \mathrm{d}\mu_\theta \quad \text{for } i_1, i_2, \dots, i_d \in \mathbb{Z}_{\geq 0}$$

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Example:

Let $\Theta = \{(a, b) \in \mathbb{R}^2 \mid a \leq b\}$ be the space of **line segments** in $\mathbb{R}$.
Let $\mu_{(a,b)}$ be the uniform probability distributions on the line segment (a, b).

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$$\Rightarrow m_i(\mu_{(a,b)}) = \frac{1}{b-a}\int_a^b w^i \,\mathrm{d}w = \frac{1}{i+1}\frac{b^{i+1} - a^{i+1}}{b-a}$$
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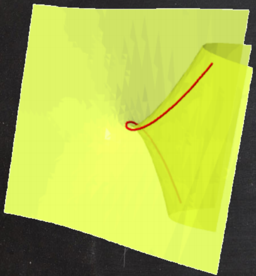
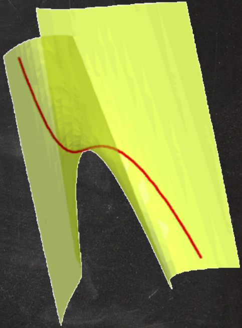
The first two moments m_1, m_2 yield two solutions (a, b),
but only one with $a \leq b$.

Example: line segments

The moments $m_1, m_2, \ldots, m_r$ of a line segment (a, b) are not algebraically independent! **The lie on a surface in $\mathbb{R}^r$.**





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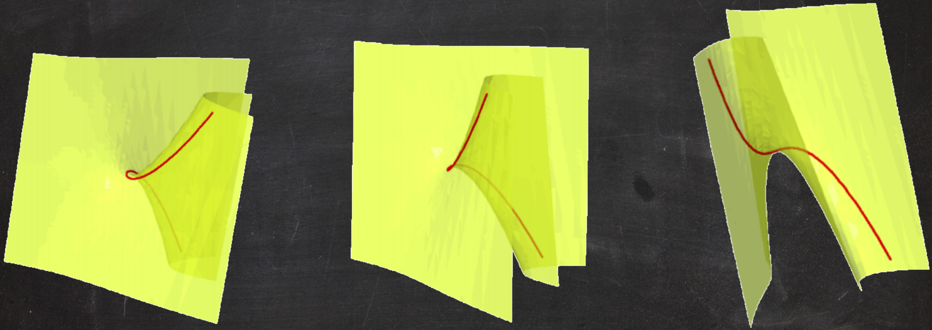

- for $r = 3$, the surface is defined by $2m_1^3 - 3m_1m_2 + m_3 = 0$
- it contains the **twisted cubic curve** corresponding to degenerate line segments (a, a) of length 0

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Practical meaning:
If the given moments have noise, we cannot recover the line segment!

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Practical meaning:
If the given moments have noise, we cannot recover the line segment! We first need to denoise the moments, i.e. find a closest point on the moment surface.

$\Rightarrow$ **We need to understand the moment surface, i.e. the algebraic dependencies among the moments**.

Example: line segments

The moment surface in $\mathbb{R}^r$ of the first r moments $m_1, m_2, \ldots, m_r$

- has degree $\binom{r}{2}$
- and its prime ideal is generated by the 3×3 minors of

$$\begin{pmatrix} 0 & 1 & 2m_1 & 3m_2 & 4m_3 & \cdots & (r-1)m_{r-2} \\ 1 & 2m_1 & 3m_2 & 4m_3 & 5m_4 & \cdots & r\, m_{r-1} \\ 2m_1 & 3m_2 & 4m_3 & 5m_4 & 6m_5 & \cdots & (r+1)m_r \end{pmatrix}.$$

- These cubics form a Gröbner basis.

Intermezzo: Optimization

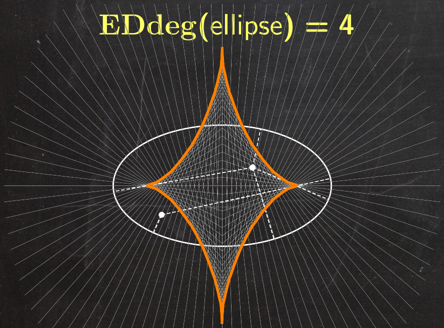
finding a closest point on an algebraic variety

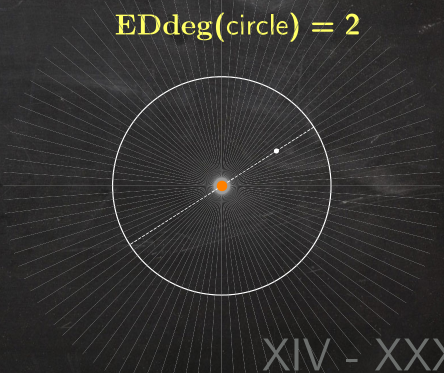
Euclidean distance degree

The **ED degree** of an algebraic variety $X \subset \mathbb{R}^n$ is the number of critical points (over $\mathbb{C}$) of the Euclidean distance

$$\begin{aligned} X &\longrightarrow \mathbb{R}, \\ x &\longmapsto \|x - u\|^2 \end{aligned}$$

between a generic point $u \in \mathbb{R}^n$ and the variety X.





back to

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Similarities to reconstruction in computer vision:

Instead of the joint camera map, we study the **moment map**

$$\begin{aligned} \Phi : \Theta &\longrightarrow \mathbb{R}^{\mathcal{I}}, \\ \theta &\longmapsto m_{i_1 i_2 \ldots i_d}(\mu_\theta), \end{aligned}$$

where $\mathcal{I} \subset \mathbb{Z}_{\geq 0}$ is a finite index set, and ask for its fibers.

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Typical settings in practice:

1. the fibers of Φ are generically finite and non-empty
   $\rightarrow$ can solve reconstruction problem for any generic input
2. $\mathrm{im}(\Phi)$ lies in a proper subvariety
   $\rightarrow$ need to denoise input before reconstructing

Example: quadrilaterals

Let $\Theta = \{\Box \subset \mathbb{R}^2\} \subset \left(\mathbb{R}^2\right)^4$ be the space of **quadrilaterals** in $\mathbb{R}^2$.
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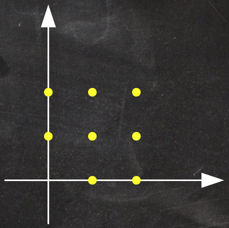
Let $\mathcal{I}$ be as shown on the right.

The fibers of $\Phi : \Theta \to \mathbb{R}^8$ are generically finite, of cardinality 80 over $\mathbb{C}$.



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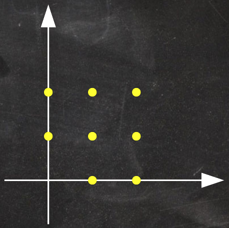
The dihedral group of order 8 acts on each fiber.
$\rightsquigarrow$ Each fiber consists of 10 "quadrilaterals".



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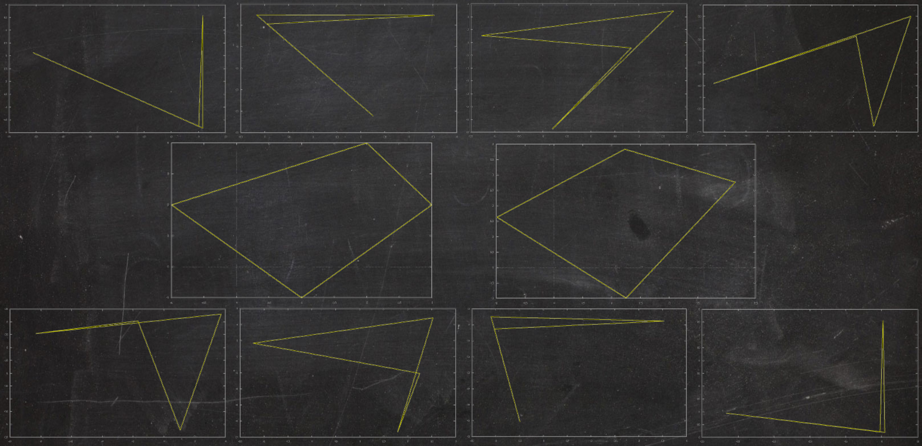

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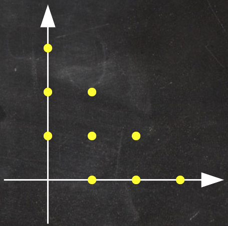

The Zariski closure of the image of $\Phi : \Theta \to \mathbb{R}^9$ is a hypersurface.

We can compute it using the **invariant ring** of the **affine group** $\mathrm{Aff}(\mathbb{R}^2)$.

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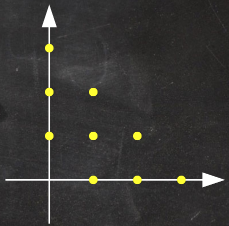

The moment hypersurface has degree 18.
Its defining polynomial has 5100 terms.

Example: quadrilaterals

```
         6   12                  3   2   12              2   4   12                  5   11                    4       11                      2   2   11              2       3   11
2125764m02 m10   - 1574640m00*m02 m03 m10   + 291600m00 m03 m10   - 25509168m01*m02 m10  m11 + 9447840m00*m02 m03*m10  m11 + 9447840m00*m01*m02 m03 m10  m11 - 3499200m00 m02*m03 m10  m11 +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
           2   4   10   2                5   10   2                      3       10   2                2       2   10   2              2   2   2   10   2              2       3   10   2
127545840m01 m02 m10  m11  + 8030664m00*m02 m10  m11  - 75582720m00*m01*m02 m03*m10  m11  - 18895680m00*m01 m02*m03 m10  m11  + 7523280m00 m02 m03 m10  m11  + 6998400m00 m01*m03 m10  m11  -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
           3   3   9   3                   4   9   3                     2   2       9   3                2   3       9   3              3   2   9   3              2           2   9   3
340122240m01 m02 m10 m11  - 80306640m00*m01*m02 m10 m11  + 226748160m00*m01 m02 m03*m10 m11  + 30443040m00 m02 m03*m10 m11  + 12597120m00*m01 m03 m10 m11  - 30093120m00 m01*m02*m03 m10 m11  -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
         3   3   9   3              4   2   8   4              2   3   8   4              2   4   8   4                  3           8   4              2       2       8   4
4665600m00 m03 m10 m11  + 510183360m01 m02 m10 m11  + 321226560m00*m01 m02 m10 m11  + 8503056m00 m02 m10 m11  - 302330880m00*m01 m02*m03*m10 m11  - 182658240m00 m01*m02 m03*m10 m11  +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
          2   2   2   8   4              3       2   8   4              5       7   5                  3   2   7   5              2       3   7   5                  4       7   5
30093120m00 m01 m03 m10 m11  + 27643680m00 m02*m03 m10 m11  - 408146688m01 m02*m10 m11  - 642453120m00*m01 m02 m10 m11  - 68024448m00 m01*m02 m10 m11  + 151165440m00*m01 m03*m10 m11  +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
           2   2           7   5              3   2       7   5              3       2   7   5              6   6   6                  4       6   6              2   2   2   6   6
365316480m00 m01 m02*m03*m10 m11  + 25894080m00 m02 m03*m10 m11  - 55287360m00 m01*m03 m10 m11  + 136048896m01 m10 m11  + 642453120m00*m01 m02*m10 m11  + 204073344m00 m01 m02 m10 m11  +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
        3   3   6   6              2   3       6   6              3               6   6              4   2   6   6                  5   5   7              2   3       5   7
186624m00 m02 m10 m11  - 243544320m00 m01 m03*m10 m11  - 103576320m00 m01*m02*m03*m10 m11  + 19215360m00 m03 m10 m11  - 256981248m00*m01 m10 m11  - 272097792m00 m01 m02*m10 m11  -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
         3       2   5   7              3   2       5   7              4           5   7              2   4   4   8              3   2       4   8              4   2   4   8
1119744m00 m01*m02 m10 m11  + 103576320m00 m01 m03*m10 m11  - 518400m00 m02*m03*m10 m11  + 136048896m00 m01 m10 m11  + 2239488m00 m01 m02*m10 m11  - 2985984m00 m02 m10 m11  +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
         4           4   8              3   3   3   9              4           3   9              5       3   9              4   2   2   10              5       2   10              5           11
1036800m00 m01*m03*m10 m11  - 1492992m00 m01 m10 m11  + 11943936m00 m01*m02*m10 m11  - 4423680m00 m03*m10 m11  - 11943936m00 m01 m10 m11   - 663552m00 m02*m10 m11   + 1327104m00 m01*m10*m11   -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
            5   11                       3       11                  2   2   2   11                2       3   11                      4   10                        2   2       10
9447840m00*m02 m10  m12 + 9447840m00*m01*m02 m03*m10  m12 + 3499200m00 m02 m03 m10  m12 - 3499200m00 m01*m03 m10  m12 + 66134880m00*m01*m02 m10  m11*m12 - 56687040m00*m01 m02 m03*m10  m11*m12 -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
          2   3       10                   2           2   10                  3   3   10                      2   3   9   2              2   4   9   2
30443040m00 m02 m03*m10  m11*m12 + 17496000m00 m01*m02*m03 m10  m11*m12 + 3499200m00 m03 m10  m11*m12 - 151165440m00*m01 m02 m10 m11 m12 - 22517352m00 m02 m10 m11 m12 +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
              3           9   2                  2       2       9   2              2   2   2   9   2              3       2   9   2                  3   2   8   3
113374080m00*m01 m02*m03*m10 m11 m12 + 137518560m00 m01*m02 m03*m10 m11 m12 - 48988800m00 m01 m03 m10 m11 m12 - 19245600m00 m02*m03 m10 m11 m12 + 75582720m00*m01 m02 m10 m11 m12 +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
          2       3   8   3                  4       8   3                  2   2           8   3              3   2       8   3              3       2   8   3
88809696m00 m01*m02 m10 m11 m12 - 75582720m00*m01 m03*m10 m11 m12 - 184757760m00 m01 m02*m03*m10 m11 m12 - 56337120m00 m02 m03*m10 m11 m12 + 80481600m00 m01*m03 m10 m11 m12 +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
              4       7   4                2   2   2   7   4              3   3   7   4              2   3       7   4              3               7   4              4   2   7   4
151165440m00*m01 m02*m10 m11 m12 + 7558272m00 m01 m02 m10 m11 m12 - 5388768m00 m02 m10 m11 m12 + 62985600m00 m01 m03*m10 m11 m12 + 59486400m00 m01*m02*m03*m10 m11 m12 - 30002400m00 m03 m10 m11 m12 -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
              5   6   5                2   3       6   5              3       2   6   5              3   2       6   5              4           6   5              2   4   5   6
151165440m00*m01 m10 m11 m12 - 375394176m00 m01 m02*m10 m11 m12 - 45349632m00 m01*m02 m10 m11 m12 + 106375680m00 m01 m03*m10 m11 m12 - 4043520m00 m02*m03*m10 m11 m12 + 370355328m00 m01 m10 m11 m12 +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
           3   2       5   6              4   2   5   6              4           5   6              3   3   4   7              4           4   7              5       4   7
246063744m00 m01 m02*m10 m11 m12 + 13281408m00 m02 m10 m11 m12 - 107205120m00 m01*m03*m10 m11 m12 - 267618816m00 m01 m10 m11 m12 - 51570432m00 m01*m02*m10 m11 m12 + 19388160m00 m03*m10 m11 m12 +
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
          4   2   3   8              5       3   8              5       2   9              6       10                  2   3   10   2              2   4   10   2
50015232m00 m01 m10 m11 m12 + 3981312m00 m02*m10 m11 m12 + 5308416m00 m01*m10 m11 m12 - 663552m00 m10*m11  m12 - 14171760m00*m01 m02 m10  m12  + 17557236m00 m02 m10  m12  -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
          2       2       10   2              2   2   2   10   2              3       2   10   2              3   2   9       2              2       3   9       2
20995200m00 m01*m02 m03*m10  m12  + 15746400m00 m01 m03 m10  m12  - 2624400m00 m02*m03 m10  m12  + 85030560m00*m01 m02 m10 m11*m12  - 49128768m00 m01*m02 m10 m11*m12  -
----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------
          2   2           9       2              3   2       9       2              3       2   9       2              4       8   2   2              2   2   2   8   2   2
```

Find distribution best explaining data

Central question:

- Let $\{\mu_\theta \mid \theta \in \Theta\}$ be a family of probability distributions.
- Let $Y = (Y_1, \ldots, Y_n)$ be n samples of observed data.

Can we find a distribution in the family that best fits the empirical data Y?

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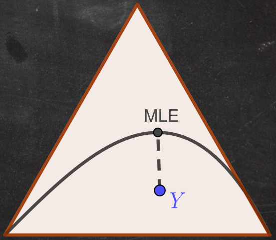
Approach: maximize the **likelihood function**

$$L_Y(\theta) := \mu_\theta(Y_1) \cdots \mu_\theta(Y_n), \qquad \text{where } \theta \in \Theta.$$



A **maximum likelihood estimate (MLE)** is a distribution in the family that maximizes the likelihood L_Y.

Example: (conditional) independence

Consider two random variables X and Y having m and n states.
Their joint probability distribution is an $m \times n$ matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}$$

whose entries are non-negative and sum to 1.

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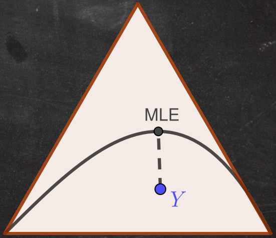
Let $\mathcal{M}_r$ be the variety of rank-r matrices in the probability simplex Δ_{mn-1}.

Matrices P in $\mathcal{M}_1$ represent **independent distributions**.



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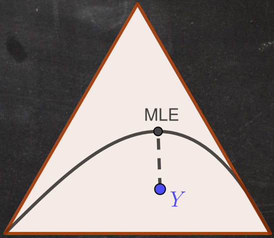

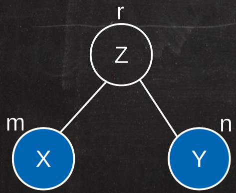

$\mathcal{M}_r$ comprises mixtures of r independent distributions.
Its elements P represent **conditionally independent distributions**.

XXI - XXX

Example: (conditional) independence

Suppose i.i.d. samples are drawn from an unknown distribution.
We summarize these data also in a matrix

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix}.$$

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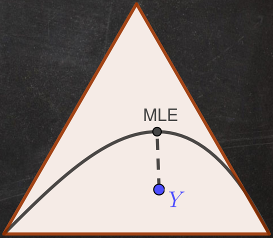
The likelihood function is the monomial

$$L_Y(P) = \prod_{i=1}^{m} \prod_{j=1}^{n} p_{ij}^{y_{ij}}.$$

An MLE for data Y is a rank-r matrix $P \in \mathcal{M}_r$ maximizing $L_Y(P)$.



ML degree

The **ML degree** of a family of distributions is the number of critical points (over $\mathbb{C}$) of the likelihood function for generic data.

some known[1] ML degrees of the rank varieties $\mathcal{M}_r$:

$(m,n) =$	$(3,3)$	$(3,4)$	$(3,5)$	$(4,4)$	$(4,5)$	$(4,6)$	$(5,5)$
$r=1$	1	1	1	1	1	1	1
$r=2$	**10**	**26**	**58**	**191**	**843**	**3119**	**6776**
$r=3$	1	1	1	**191**	**843**	**3119**	**61326**
$r=4$				1	1	1	**6776**
$r=5$							1

[1] Hauenstein, Rodriguez, Sturmfels: *Maximum likelihood for matrices with rank constraints*, Journal of Algebraic Statistics **5** (2014) 18–38.

Machine Learning

Neural networks

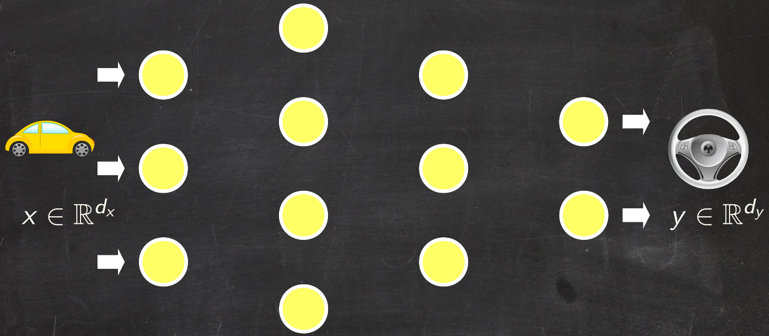
XXIV - XXX

Neural networks

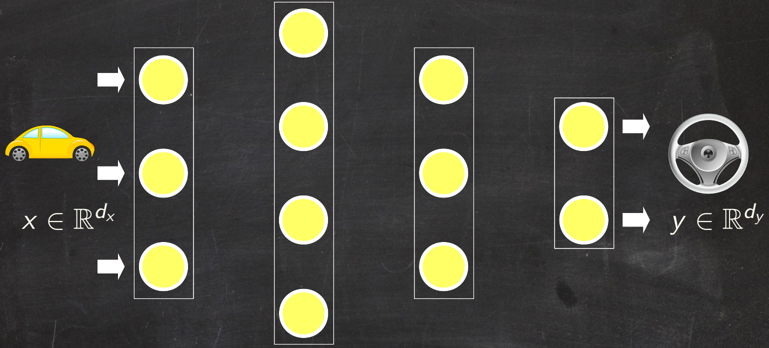
$x \in \mathbb{R}^{d_x}$

$y \in \mathbb{R}^{d_y}$

Neural networks



Neural networks



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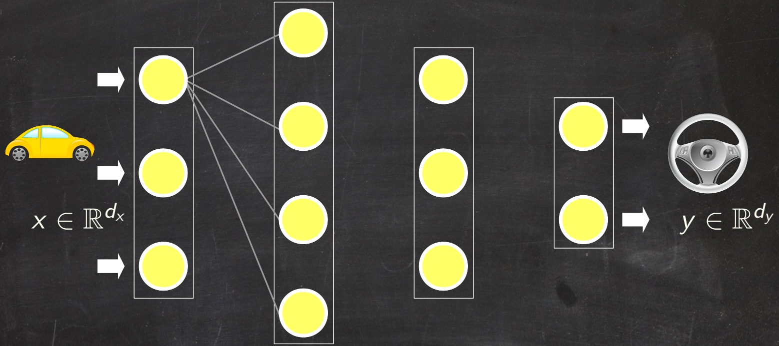
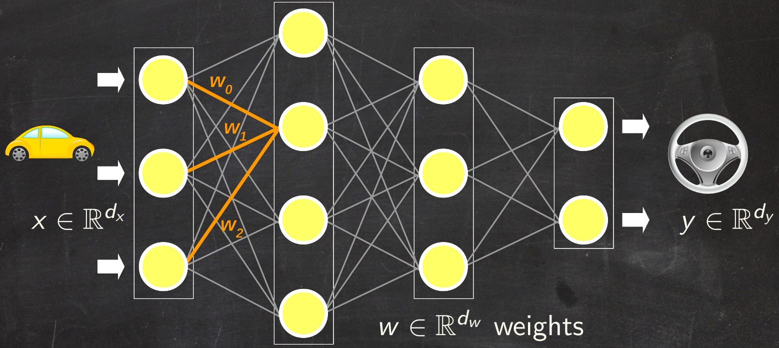
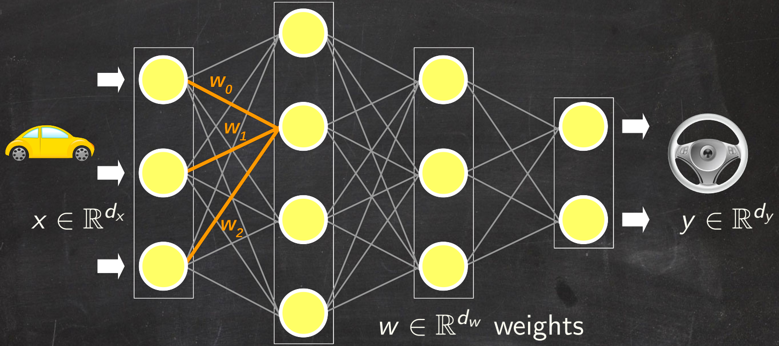


Neural networks

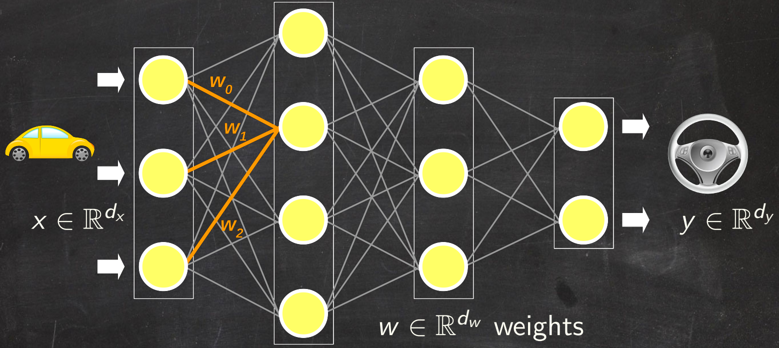


Neural networks



A neural network is defined by a continuous mapping $\Phi : \mathbb{R}^{d_w} \times \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_y}$.

Neural networks



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Definition $\mathcal{M}_\Phi := \left\{ \Phi(w, \cdot) : \mathbb{R}^{d_x} \to \mathbb{R}^{d_y} \mid w \in \mathbb{R}^{d_w} \right\}$

is called the **neuromanifold** of Φ.

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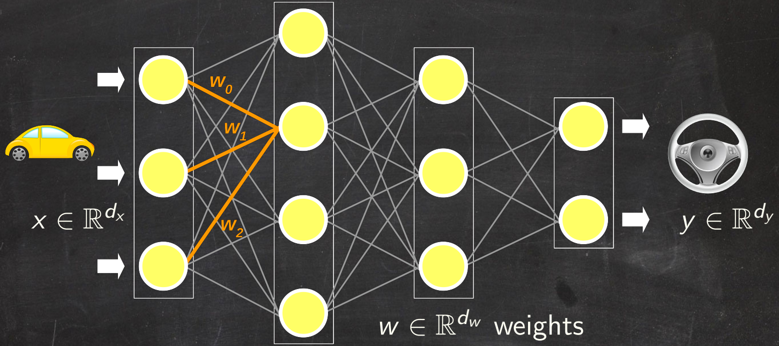

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Observation 1. Φ piecewise smooth $\Rightarrow \mathcal{M}_\Phi$ manifold with singularities

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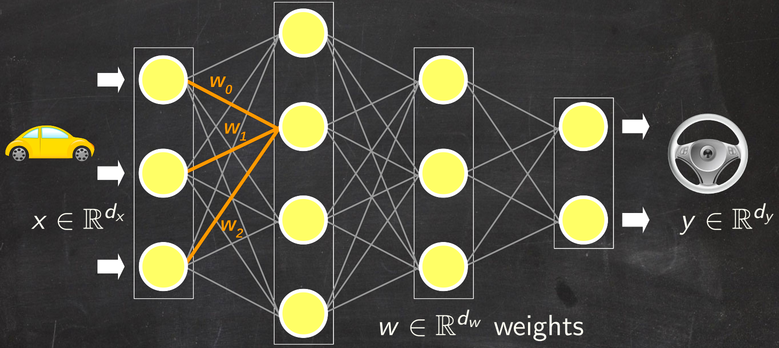

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2. $\dim \mathcal{M}_\Phi \leq d_w$

Linear networks

A **linear network** is defined by a map $\Phi : \mathbb{R}^{d_w} \times \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_y}$ of the form

$$\Phi(w, x) = W_h W_{h-1} \dots W_1 x,$$
$$\text{where } w = (W_h, \dots, W_1) \text{ and } W_i \in \mathbb{R}^{d_i \times d_{i-1}},$$

(so $d_w = d_h d_{h-1} + \ldots + d_1 d_0$, $d_x = d_0$ and $d_y = d_h$).

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Example
The neuromanifold of the linear network Φ is the bounded rank variety

$$\mathcal{M}_\Phi = \left\{ M \in \mathbb{R}^{d_h \times d_0} \mid \mathrm{rk}(M) \leq \underbrace{\min\{d_0, d_1, \dots, d_h\}}_{=:r} \right\}.$$

Loss landscapes

A **loss function** on a neural network $\Phi : \mathbb{R}^{d_w} \times \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_y}$ is of the form

$$\begin{aligned} L : \mathbb{R}^{d_w} &\xrightarrow{\quad \mu \quad} \mathcal{M}_\Phi \xrightarrow{\quad \ell|_{\mathcal{M}_\Phi} \quad} \mathbb{R}, \\ w &\longmapsto \Phi(w, \cdot) \end{aligned}$$

where ℓ is a functional defined on a subset of $C(\mathbb{R}^{d_x}, \mathbb{R}^{d_y})$ containing $\mathcal{M}_\Phi$.

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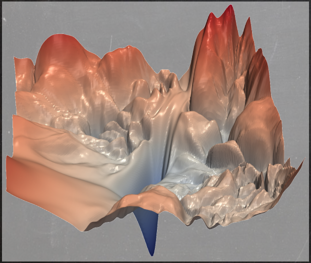
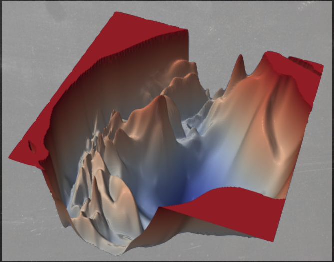


Visualizations of L



Source: Li, Hao, et al. "Visualizing the loss landscape of neural nets." Advances in Neural Information Processing Systems. 2018.

Quadratic loss on linear networks

Fixed data matrices $X \in \mathbb{R}^{d_0 \times s}$ and $Y \in \mathbb{R}^{d_h \times s}$ define a **quadratic loss**

$$\begin{aligned} \ell_{X,Y} : \mathbb{R}^{d_h \times d_0} &\longrightarrow \mathbb{R}, \\ M &\longmapsto \|MX - Y\|_F^2 \end{aligned}$$

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$$\ell_{X,Y}(M) = \|M - YX^T\|_F^2 + \text{const.}$$

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Minimizing $\ell_{X,Y}$ on the bounded rank variety $\mathcal{M}_\Phi = \{M \mid \mathrm{rk}(M) \leq r\}$ is equivalent to minimizing the Euclidean distance of YX^T to $\mathcal{M}_\Phi$.

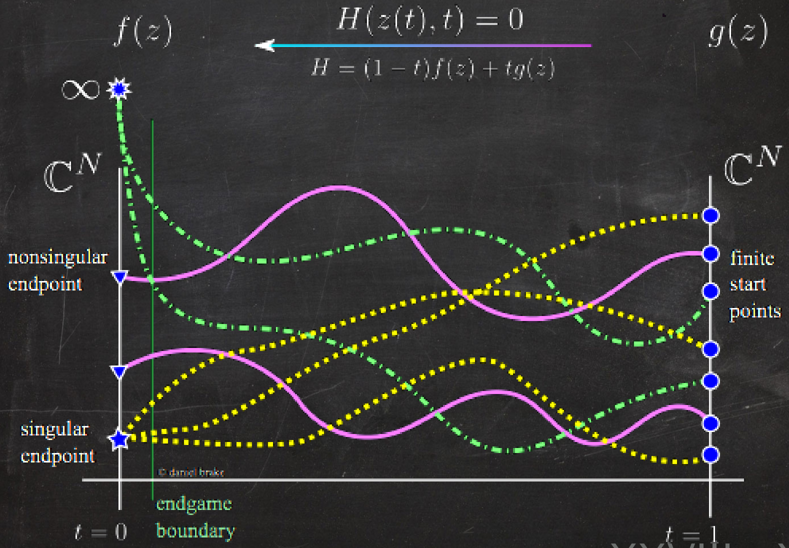
How to

solve systems of polynomial equations?

(besides Gröbner bases)

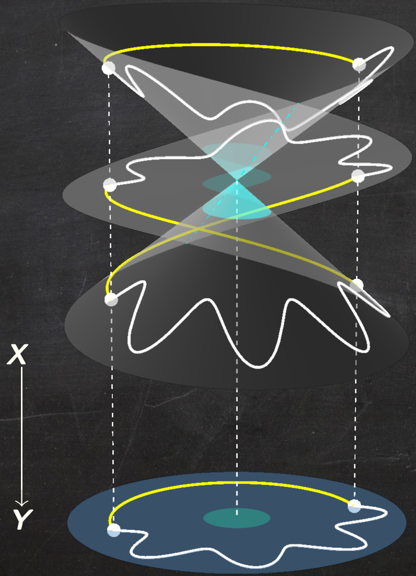
Numerical algebraic geometry

homotopy continuation



Numerical algebraic geometry

monodromy



Numerical algebraic geometry

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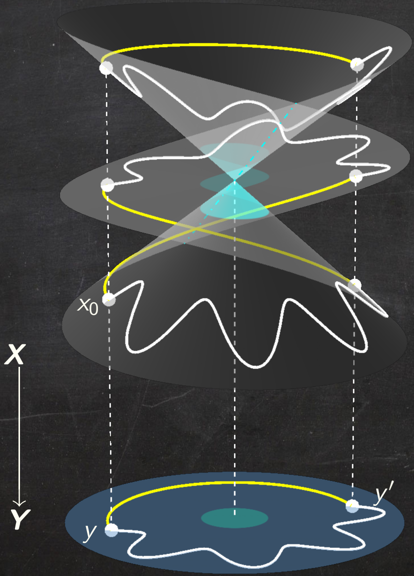

Numerical algebraic geometry

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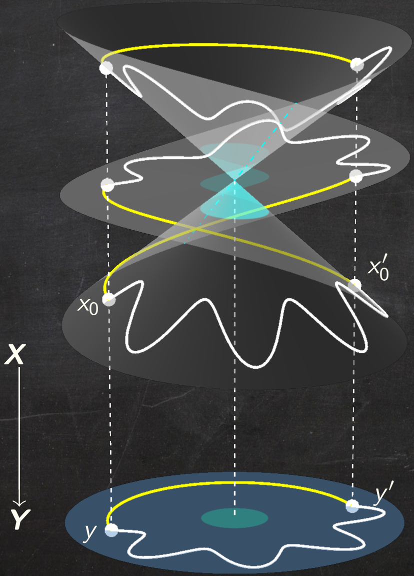

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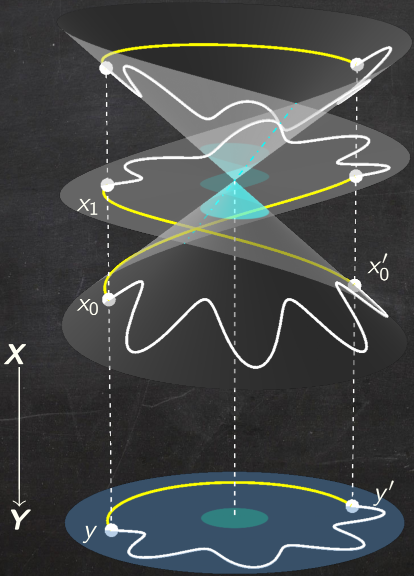

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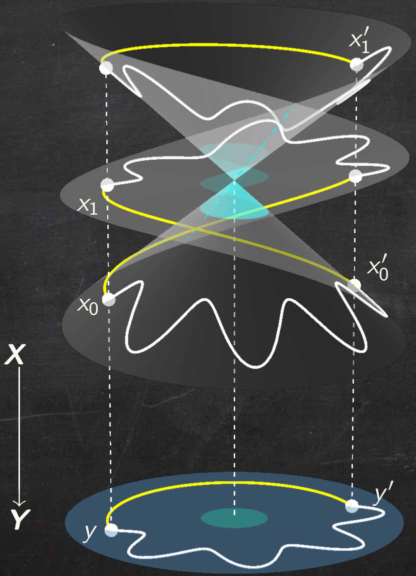

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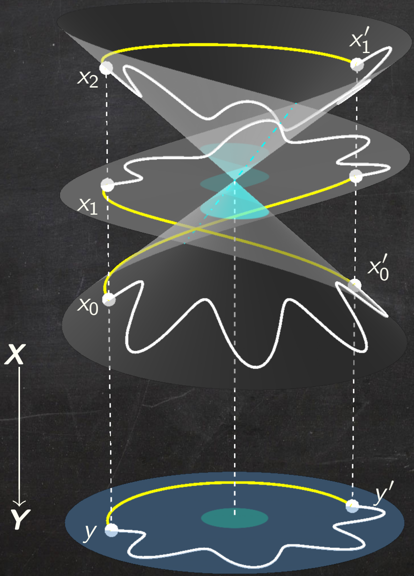
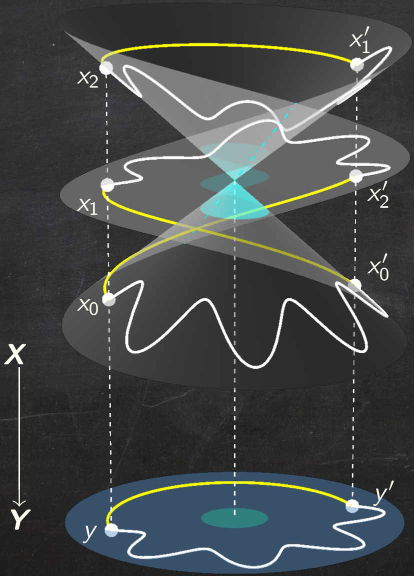


Numerical algebraic geometry

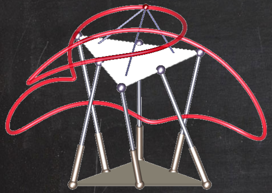
monodromy



The world is non-linear!

Application areas

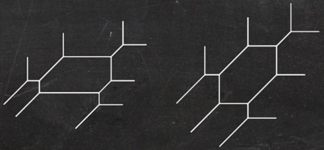
- computer vision
- algebraic statistics
- machine learning
- optimization
- robotics



- complexity theory
- biochemistry
- music
- ...

Toolbox

- algebraic geometry
- combinatorics
- convex and discrete geometry
- representation theory
- symbolic and numerical computations
- tropical geometry



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