Changing Views on Curves and Surfaces

Kathlén Kohn (ICERM, Brown University)

joint work with Bernd Sturmfels (MPI Leipzig, UC Berkeley) and Matthew Trager (NYU)

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Visual Event Surface

Consider a fixed curve or surface in 3-space.
Take pictures of that object with a moving camera.

At some camera points the image undergoes a qualitative change.
These points form the visual event surface.

Section 1

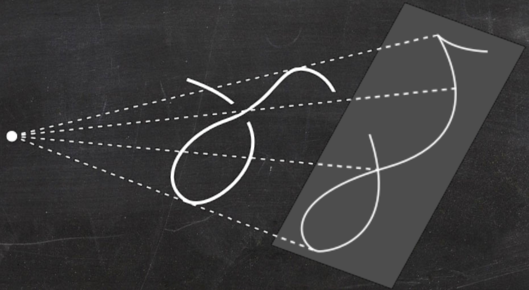
Curves

Visual Event Surface

Consider a smooth curve in 3-space

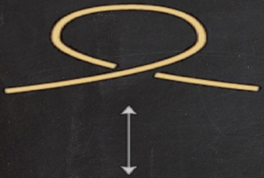
- that is not contained in any plane, and
- has degree d and genus g.

Projection from a general camera point yields a plane curve with $\frac{1}{2}(d-1)(d-2)-g$ nodes (over $\mathbb{C}$), and no other singularities.



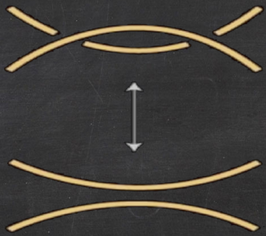
The visual event surface consists of those camera points where the plane curve has a different singularity structure.

Visual Event Surface: 3 Components



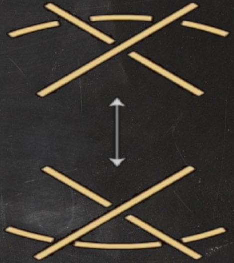
Tangential surface
union of all tangent lines to the curve

$\rightsquigarrow$ cusp in image



Edge surface
union of lines spanned by 2 points on curve whose tangent lines lie in a common plane

$\rightsquigarrow$ tacnode in image



Trisecant surface
union of lines passing through 3 points on curve

$\rightsquigarrow$ triple point in image

Coisotropic Hypersurfaces

There are 2 coisotropic hypersurface associated to a curve C in $\mathbb{P}^3$:

- dual surface $C^\vee$ in $(\mathbb{P}^3)^*$: tangent planes to C,
- Chow hypersurface $\mathrm{Ch}(C)$ in $\mathrm{Gr}(1, \mathbb{P}^3)$: lines meeting C.

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Their (iterated) singular loci yield the 3 components of the visual event surface of C:

$\mathrm{Gr}(1, \mathbb{P}^3)$
$mult. = 0$

$\mathrm{Ch}(C)$
$mult. = 1$

{secant lines to C}
$mult. = 1 + 1$

{trisecant lines}
$mult. = 1 + 1 + 1$

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Their (iterated) singular loci yield the 3 components of the visual event surface of C:

$(\mathbb{P}^3)^*$
mult. $= 1$

$C^\vee$
mult. $= 2$

dual curve to tang. surface
mult. $= 3$

dual curve to edge surface
mult. $= 2 + 2$

$\mathrm{Gr}(1,\mathbb{P}^3)$
mult. $= 0$

$\mathrm{Ch}(C)$
mult. $= 1$

{secant lines to C}
mult. $= 1 + 1$

{trisecant lines}
mult. $= 1 + 1 + 1$

Degrees

For a general space curve C of degree d and genus g, the degrees of the compononents of its visual event surface are

$$\begin{aligned} \text{tangential surface} &: \quad 2(d+g-1), \\ \text{edge surface} &: \quad 2(d-3)(d+g-1), \\ \text{trisecant surface} &: \quad \frac{(d-1)(d-2)(d-3)}{3} - (d-2)g. \end{aligned}$$

d	g	tangential surface	edge surface	trisecant surface
3	0	4	0	0
4	0	6	6	2
4	1	8	8	0
5	0	8	16	8
5	1	10	20	5
5	2	12	24	2
6	0	10	30	20
6	1	12	36	16
6	2	14	42	12
6	3	16	48	8
6	4	18	54	4

Section 2

Surfaces

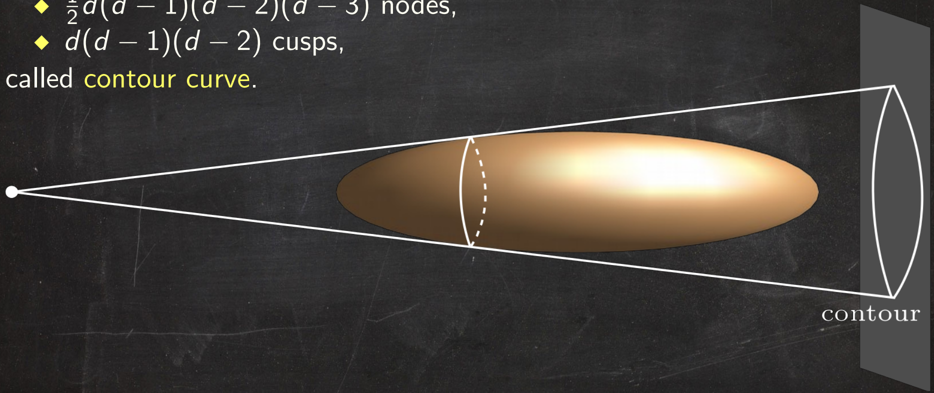
Visual Event Surface

Consider a general surface in 3-space of degree d.

The branch locus of the projection from a general point is a plane curve with

- degree $d(d-1)$,
- $\frac{1}{2}d(d-1)(d-2)(d-3)$ nodes,
- $d(d-1)(d-2)$ cusps,

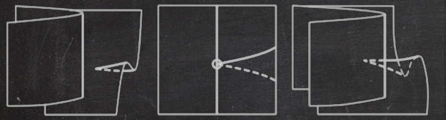
called contour curve.



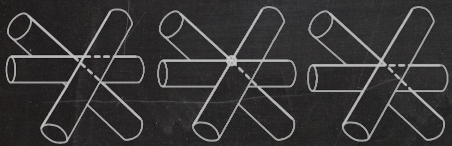
The visual event surface consists of those camera points where the contour curve has a different singularity structure.

Visual Event Surface: 5 Components

Edge surface

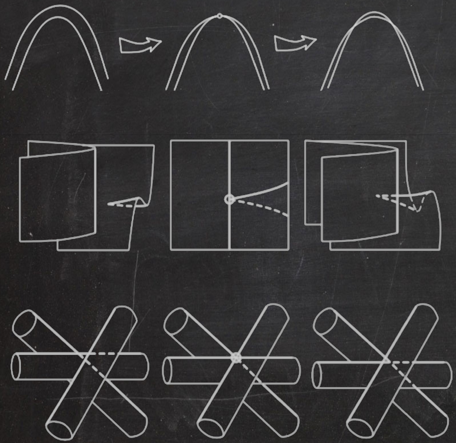


Cusp crossing surface



Tritangent surface

Visual Event Surface: 5 Components



Edge surface

union of bitangent lines contained in bitangent planes

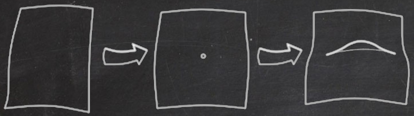
Cusp crossing surface

union of lines with contact of order $3 + 2$ at 2 points of the surface

Tritangent surface

union of all tritangent lines to the surface

Visual Event Surface: 5 Components

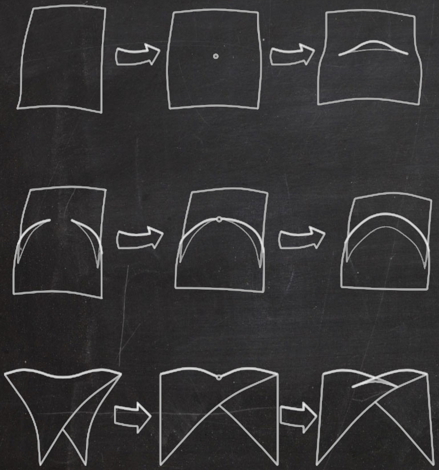


Parabolic surface

Over $\mathbb{R}$ there are 2 possible singularities in the contour curve.

Flecnodal surface

Visual Event Surface: 5 Components



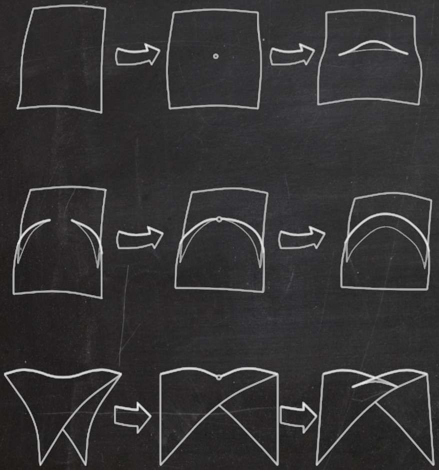
Parabolic surface

A general point on the surface has 2 lines with contact of order 3. A point is called parabolic if there is just 1 such line.

Over $\mathbb{R}$ there are 2 possible singularities in the contour curve.

Flecnodal surface

Visual Event Surface: 5 Components



Parabolic surface

union of lines with contact of order 3 at a parabolic point of the surface

> A general point on the surface has 2 lines with contact of order 3. A point is called parabolic if there is just 1 such line.

Over $\mathbb{R}$ there are 2 possible singularities in the contour curve.

Flecnodal surface

union of lines with contact of order 4 at a point of the surface

Coisotropic Hypersurfaces

There are 2 coisotropic hypersurface associated to a general surface S in $\mathbb{P}^3$:

- dual surface $S^\vee$ in $(\mathbb{P}^3)^*$: tangent planes to S,
- Hurwitz hypersurface $\mathrm{Hur}(S)$ in $\mathrm{Gr}(1, \mathbb{P}^3)$: tangent lines to S.

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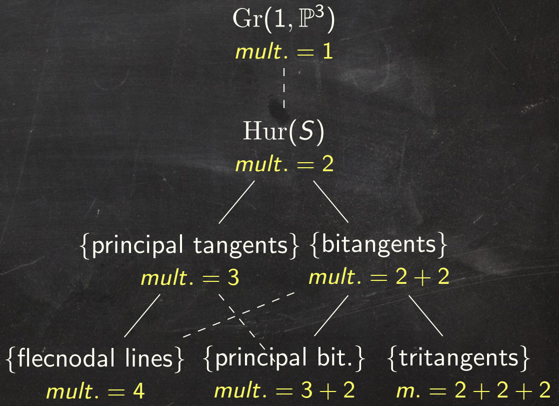

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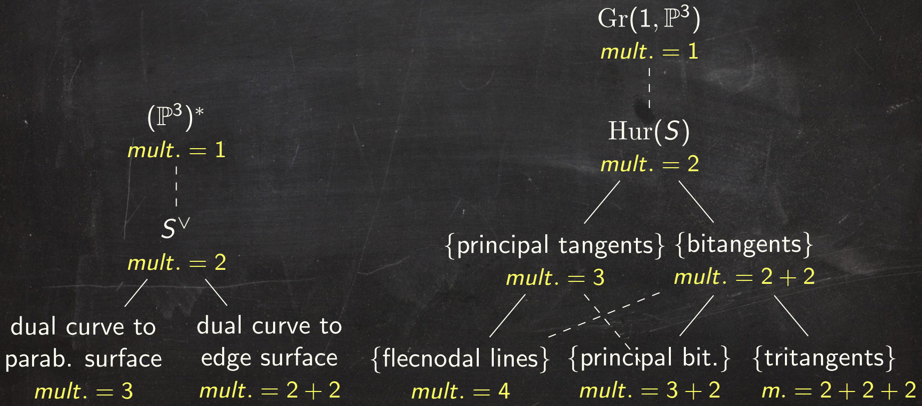
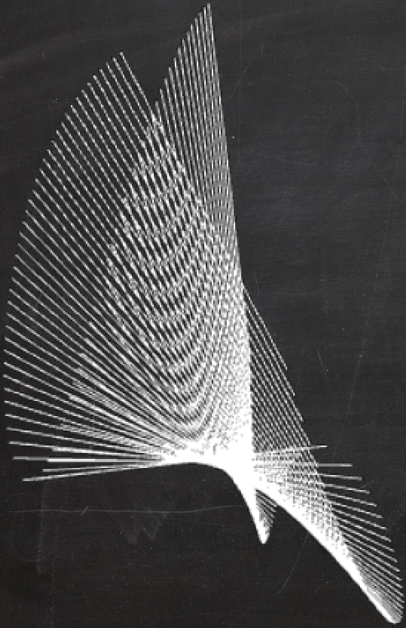


Degrees

For a general surface S in $\mathbb{P}^3$ of degree d, the degrees of the compononents of its visual event surface are

$$\begin{aligned}
&\text{flecnodal surface} &:&\quad 2d(d-3)(3d-2),\\
&\text{cusp crossing surface} &:&\quad d(d-3)(d-4)(d^2+6d-4),\\
&\text{tritangent surface} &:&\quad \tfrac{1}{3}d(d-3)(d-4)(d-5)(d^2+3d-2),\\
&\text{edge surface} &:&\quad d(d-2)(d-3)(d^2+2d-4),\\
&\text{parabolic surface} &:&\quad 2d(d-2)(3d-4).
\end{aligned}$$

d	flecnodal	cusp crossing	tritangent	edge	parabolic
3	0	0	0	0	30
4	80	0	0	160	128
5	260	510	0	930	330
6	576	2448	624	3168	672
7	1064	7308	3808	8260	1190



Thanks for your attention