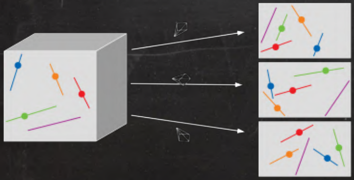
Nonlinear Algebra & Minimal Problems in Computer Vision

Kathlén Kohn

KTH VETENSKAP OCH KONST
WASP WALLENBERG AI, AUTONOMOUS SYSTEMS AND SOFTWARE PROGRAM



SFSG SWEDISH FOUNDATIONS' STARTING GRANT
Ragnar Söderbergs STIFTELSE
Göran Gustafssons Stiftelser

Linear algebra

All undergraduate students learn about Gaussian elimination, a general method for solving linear systems of algebraic equations:

Input:

$$x + 2y + 3z = 5$$
$$7x + 11y + 13z = 17$$
$$19x + 23y + 29z = 31$$

Output:

$$x = -35/18$$
$$y = 2/9$$
$$z = 13/6$$

Solving very large linear systems is central to applied mathematics.

Nonlinear algebra

Lucky students also learn about Gröbner bases, a general method for non-linear systems of algebraic equations:

Input:

$$x^2 + y^2 + z^2 = 2$$
$$x^3 + y^3 + z^3 = 3$$
$$x^4 + y^4 + z^4 = 4$$

Output:

$$3z^{12} - 12z^{10} - 12z^9 + 12z^8 + 72z^7 - 66z^6 - 12z^4 + 12z^3 - 1 = 0$$

$$4y^2 + (36z^{11} + 54z^{10} - 69z^9 - 252z^8 - 216z^7 + 573z^6 + 72z^5 - 12z^4 - 99z^3 + 10z + 3)\, y + 36z^{11} + 48z^{10} - 72z^9 - 234z^8 - 192z^7 + 564z^6 - 48z^5 + 96z^4 - 96z^3 + 10z^2 + 8 = 0$$

$$4x + 4y + 36z^{11} + 54z^{10} - 69z^9 - 252z^8 - 216z^7 + 573z^6 + 72z^5 - 12z^4 - 99z^3 + 10z + 3 = 0$$

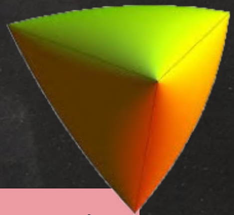
This is very hard for large systems, but . . .

The world is non-linear!

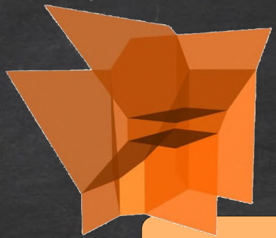
Many models in the sciences and engineering are characterized by polynomial equations. Such a set is an algebraic variety.

- Algebraic statistics
- Machine learning
- Optimization
- **Computer vision**
- Robotics
- Complexity theory
- Cryptography
- Biology
- Economics
- ...

Nonlinear Algebra



Algebraic Geometry +



Combinatorics

Convex Geometry

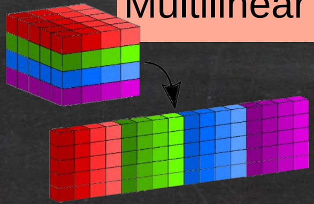
Discrete Geometry

Tropical Geometry

Algebraic Topology

Representation Theory

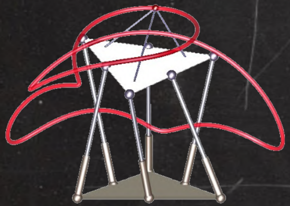
Multilinear Algebra



Numerics

...

Number Theory



Applications

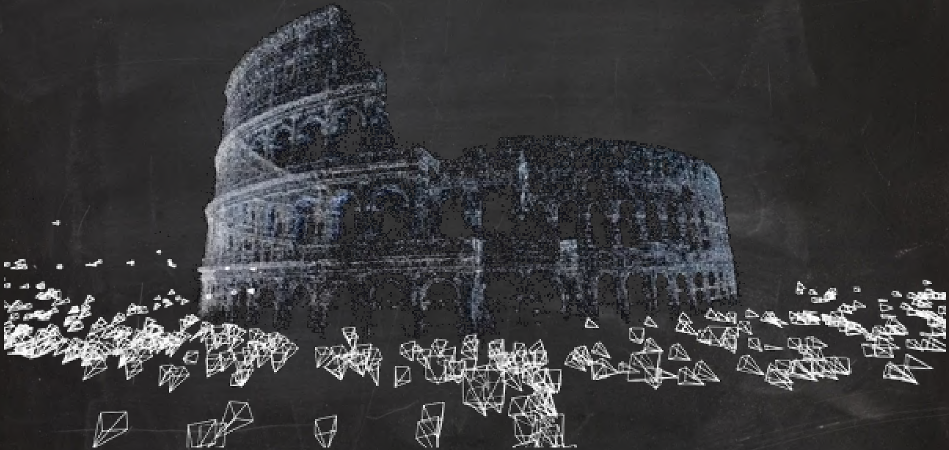
3D reconstruction

2d pictures

given images taken by unknown cameras, want to recover

3d modell

Reconstruct 3D scenes and camera poses from 2D images

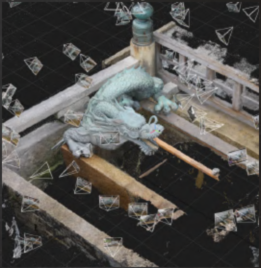


Rome in a Day: S. Agarwal, Y. Furukawa, N. Snavely, I. Simon, S. Seitz, R. Szeliski

3D reconstruction pipeline

Input:
2D images

Output:
3D scene & cameras



3D reconstruction pipeline

Input:
2D images

Image
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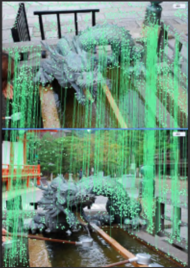
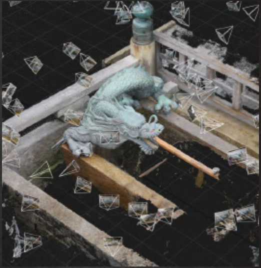


Identify common
points and lines
on given images

Output:
3D scene & cameras



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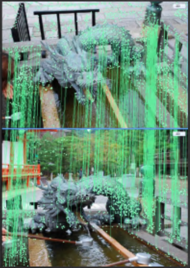
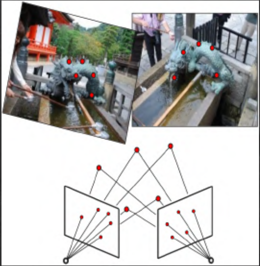


Identify common points and lines on given images

Algebraic reconstruction



Reconstruct 3D points and lines & camera poses

Output:
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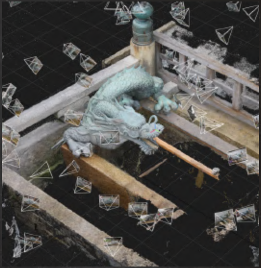

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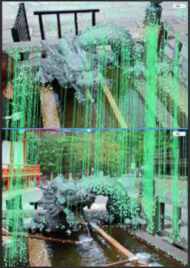

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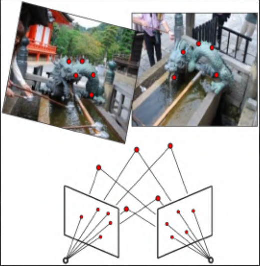
**Algebraic
reconstruction**

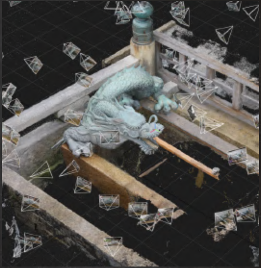


Reconstruct 3D points and
lines & camera poses

$\Updownarrow$

nonlinear inverse problem

Output:
3D scene & cameras



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RANSAC (RANdom SAmple Consensus) provides robust estimation !

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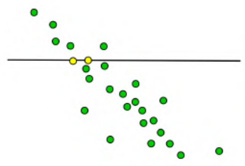
1) Randomly select a subset of the data
2) Fit a model to the selected subset
3) Determine the number of outliers
4) Repeat steps 1-3 to find a consensus (& outliers)

Example: fitting a line to points



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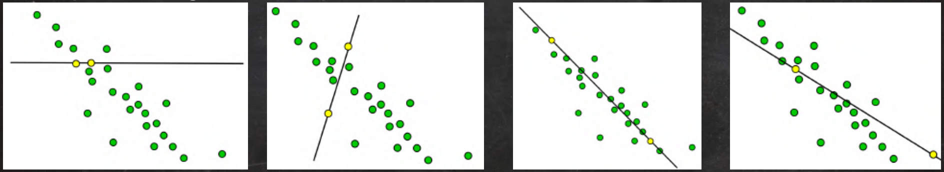

few outliers!

Observations are often noisy, and can even be corrupted with outliers.
RANSAC (RANdom SAmple Consensus) provides robust estimation !

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2d pictures

$\longrightarrow$

3d modell

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2d pictures

$\longrightarrow$

3d modell

for general algebraic inverse problems, step 2) means to solve a system of polynomial equations!

need to do this very fast, say in < 1 **ms!** (due to step 4))

Which polynomial systems can be solved fast?

Typically, the systems of polynomial equations we can solve the fastest are those whose solution sets are generically 0-dimensional (i.e., non-empty and finite)

– these are called **minimal problems** in computer vision–

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Typically, the systems of polynomial equations we can solve the fastest are those whose solution sets are generically 0-dimensional (i.e., non-empty and finite)

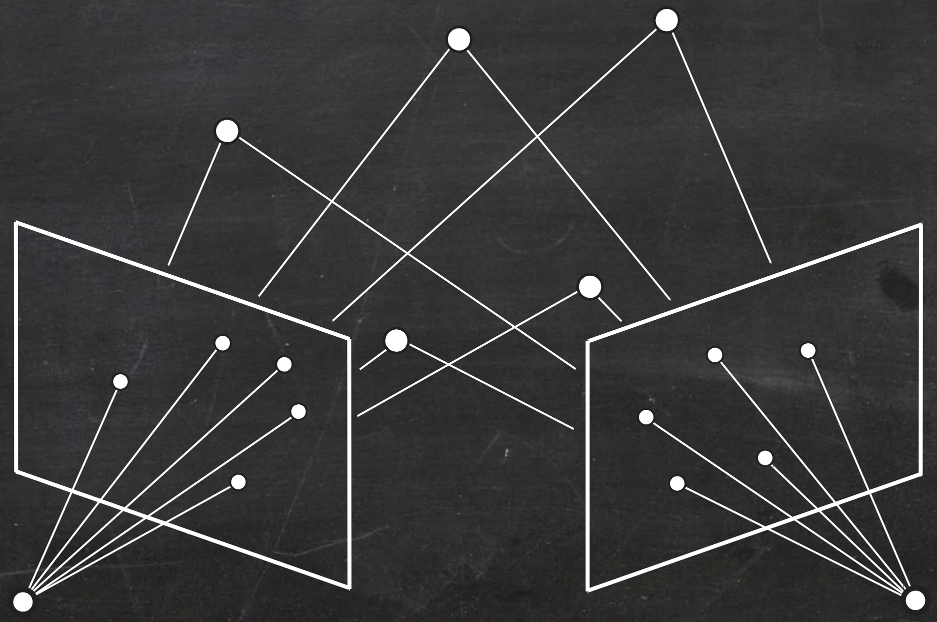
– these are called **minimal problems** in computer vision–

and whose solutions sets have small cardinality.

– known as the **degree** of the minimal problem

Example: The 5-Point Problem

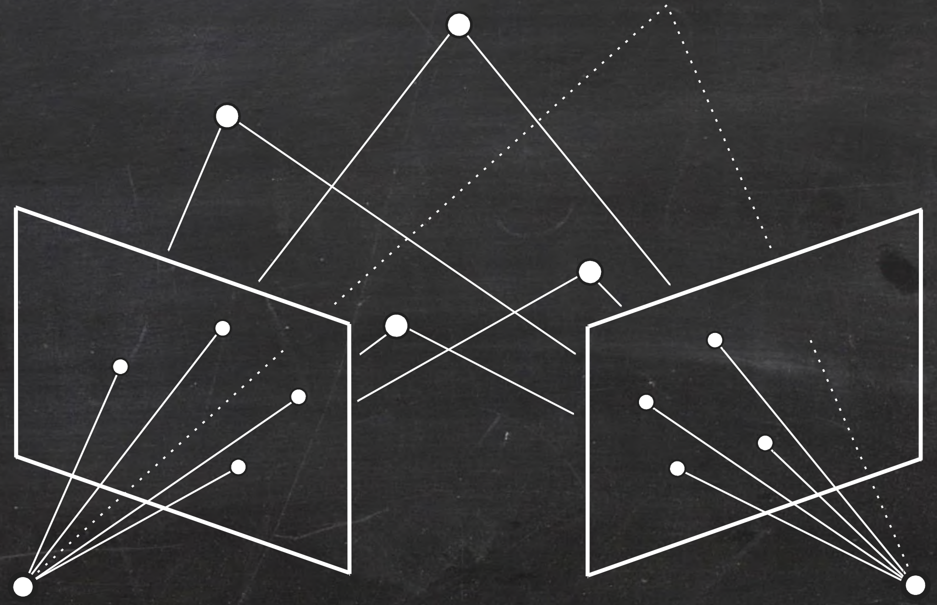
- Given: 2 images showing **5** points
- Goal: recover **5** points in 3D, and both (relative) camera poses



This problem has 20 solutions for generic input images (counted over the complex numbers).

An Underconstrained Problem

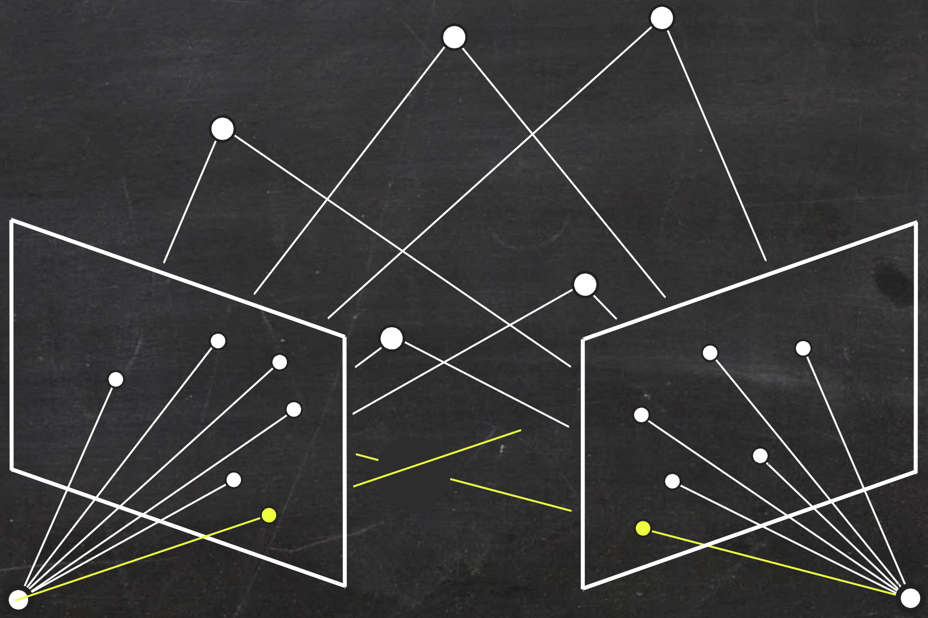
- Given: 2 images showing **4** points
- Goal: recover **4** points in 3D, and both (relative) camera poses



This problem has infinitely many solutions for generic input images.

An Overconstrained Problem

- Given: 2 images showing **6** points
- Goal: recover **6** points in 3D, and both (relative) camera poses



This problem has 0 solutions for generic input images.

Some input images have solutions, but they are **not stable under noise** in the input images!

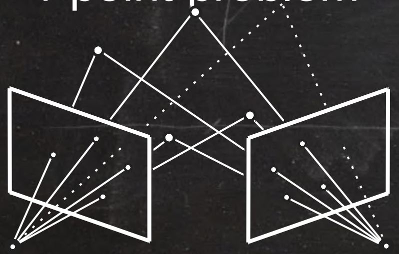
Minimal Problems

Definition: A 3D reconstruction problem is **minimal** if

$$0 < \#\ \text{solutions} < \infty$$

for generic (random) input images.

4-point problem



∞ solutions
not minimal

5-point problem

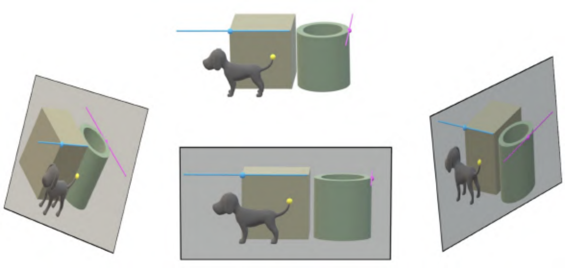
20 solutions
minimal

6-point problem

0 solutions
not minimal

another minimal problem

Given: point, point on line & point on line on each 2d-image
Goal: compute point, point on line & point on line in 3-space, and the three (relative) camera poses

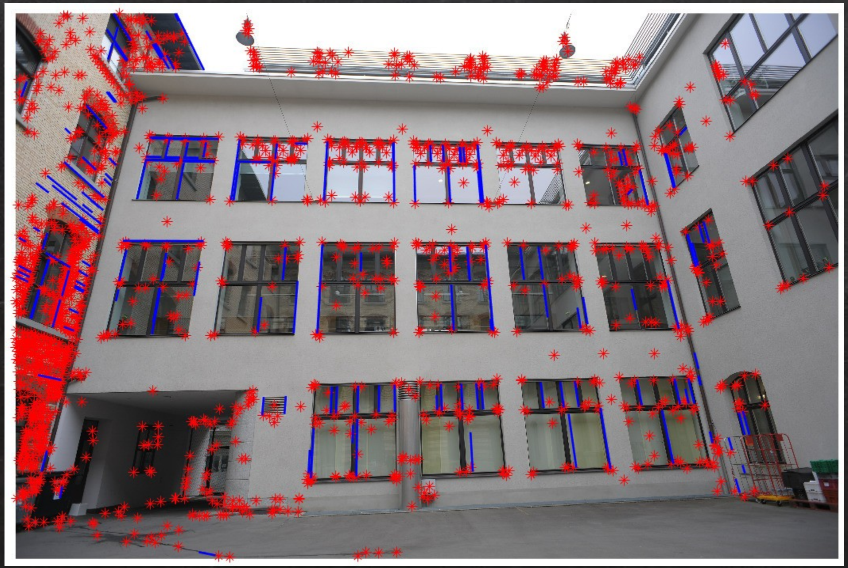
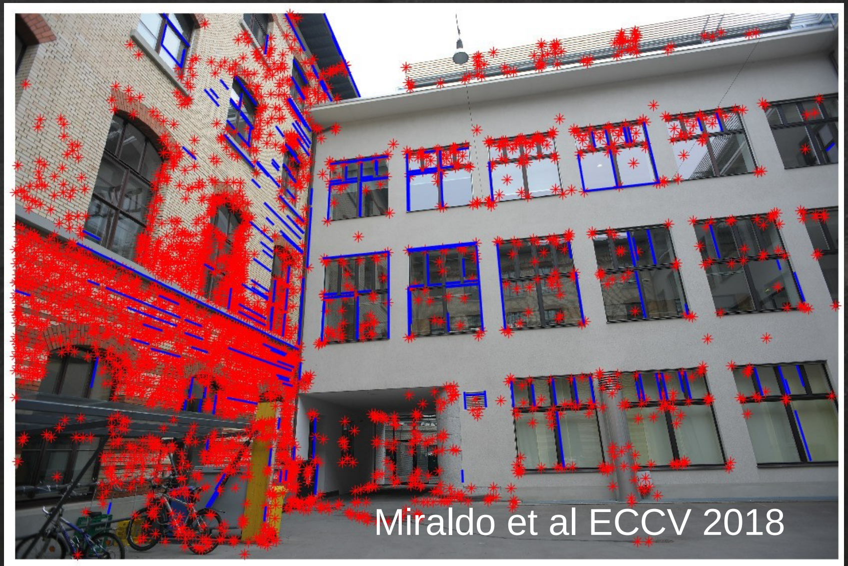


Generally has 312 complex solutions.

Fundamental Research Questions

1. Can we list **all** minimal problems?
2. How many solutions do they have?

We do not only want to work with points,
but also with lines and their incidences!

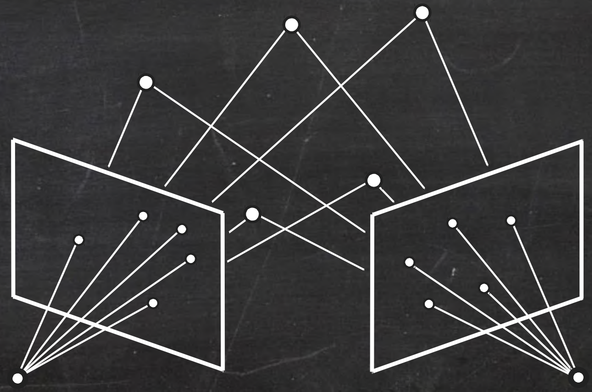



Miraldo et al ECCV 2018

Our Result

T. Duff, K. Kohn, A. Leykin, T. Pajdla | ICCV 2019

We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image. for calibrated cameras.



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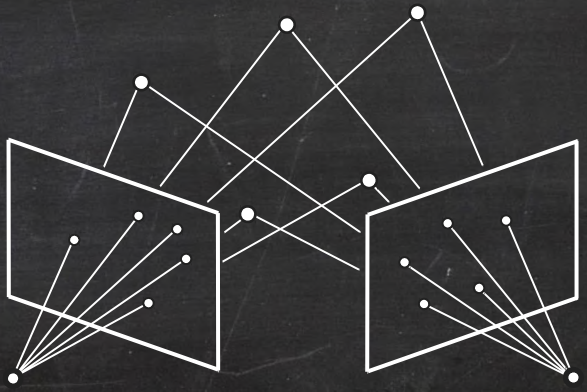
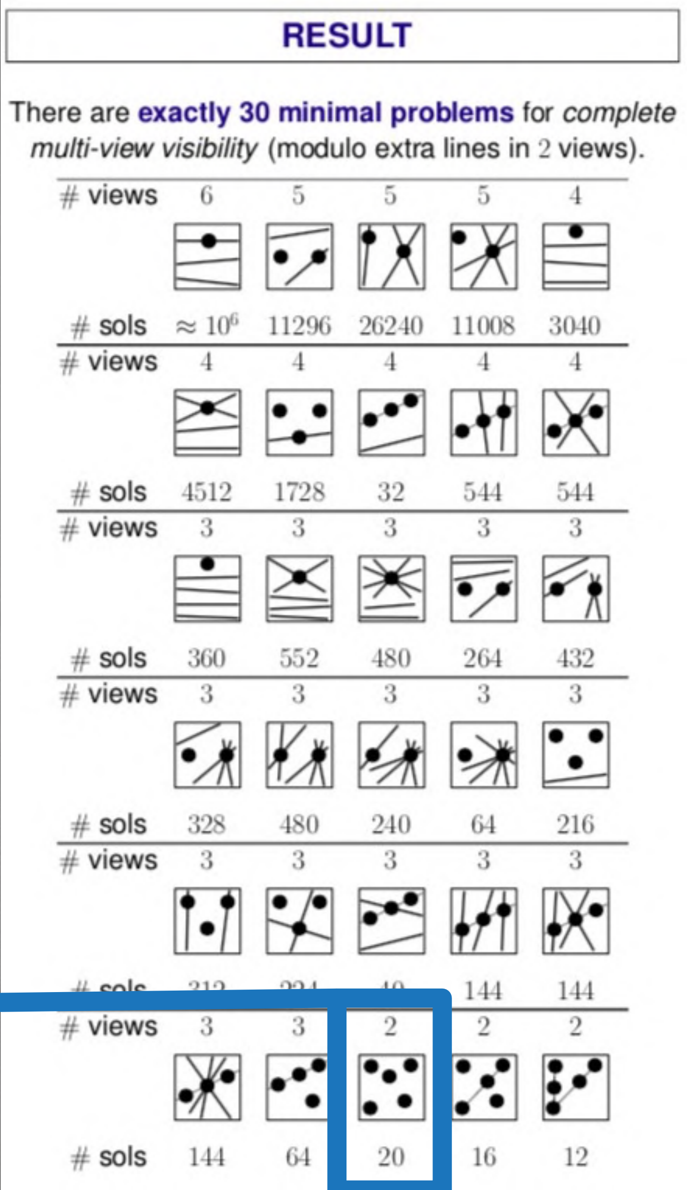


The 5-point problem has 20 solutions.



RESULT

There are **exactly 30 minimal problems** for *complete multi-view visibility* (modulo extra lines in 2 views).

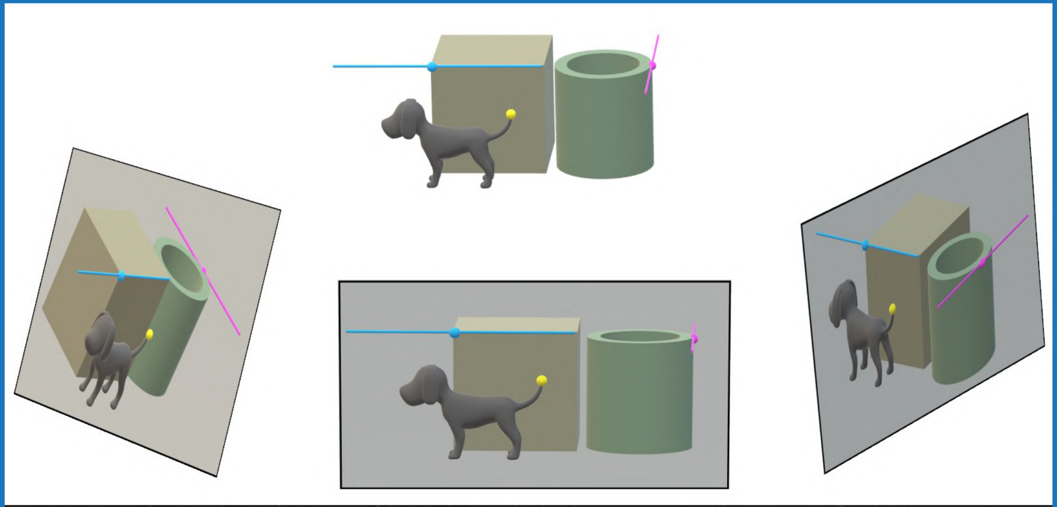
# views	6	5	5	5	4
# sols	≈ 10^6	11296	26240	11008	3040
# views	4	4	4	4	4
# sols	4512	1728	32	544	544
# views	3	3	3	3	3
# sols	360	552	480	264	432
# views	3	3	3	3	3
# sols	328	480	240	64	216
# views	3	3	3	3	3
# sols	312	224	40	144	144
# views	3	3	2	2	2
# sols	144	64	20	16	12

Our Result

We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image.

First solver for such a high-degree problem based on state-of-the-art algorithms from **numerical algebraic geometry:**

TRPLP – Trifocal Relative Pose from Lines at Points, Fabbri et. al., CVPR 2020



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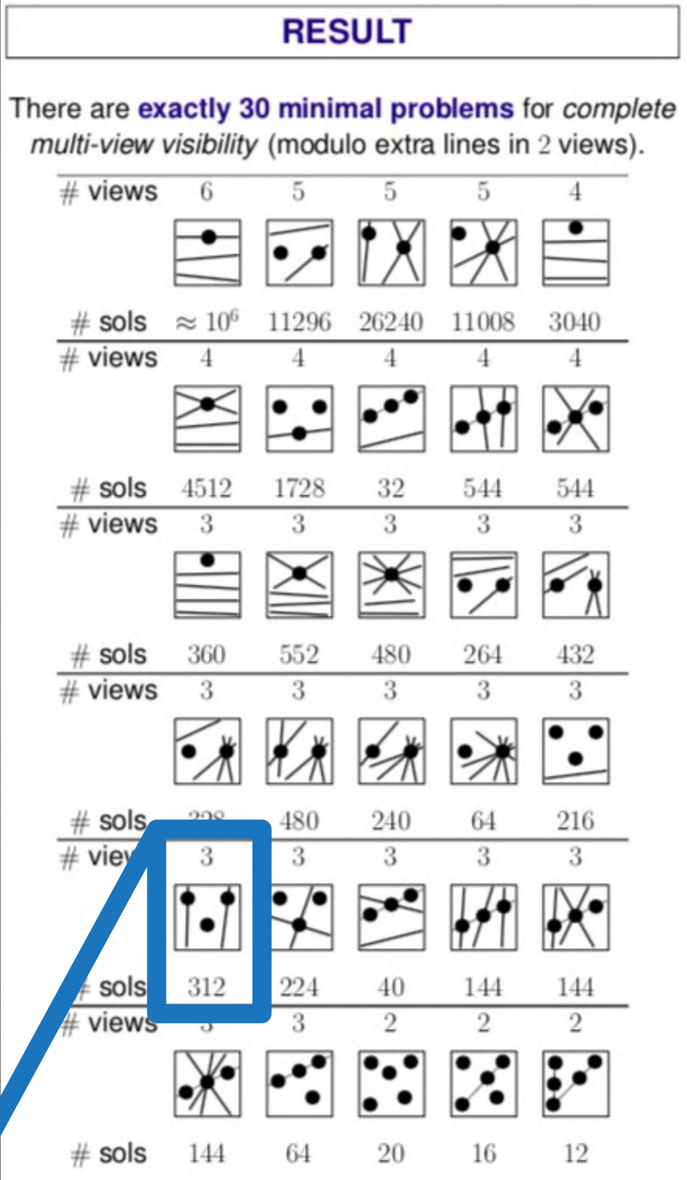

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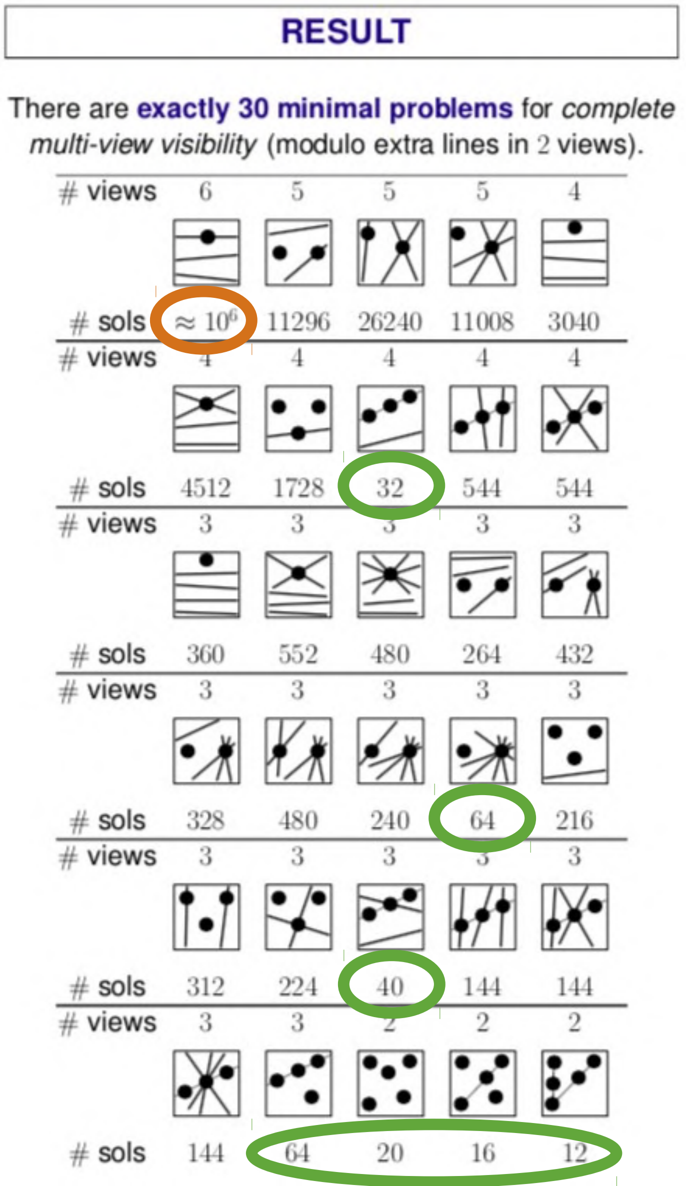
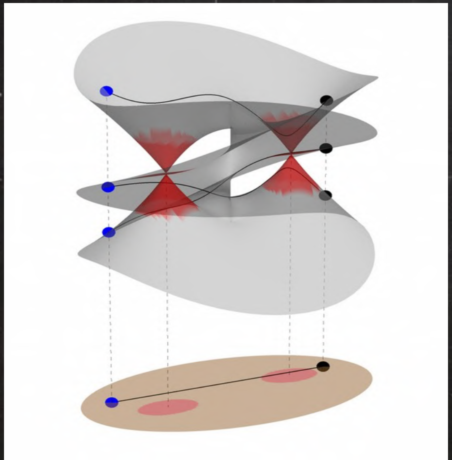
We **measure the complexity of each minimal problem** by computing its number of solutions

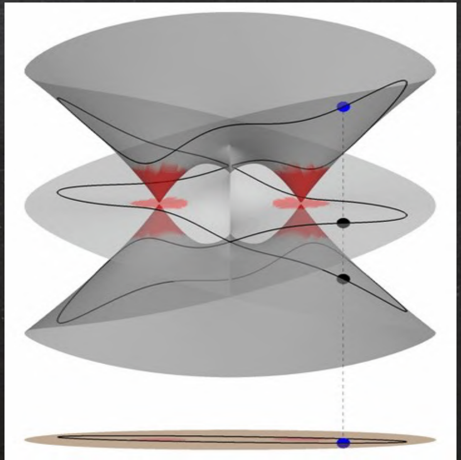
(counted over the complex numbers).

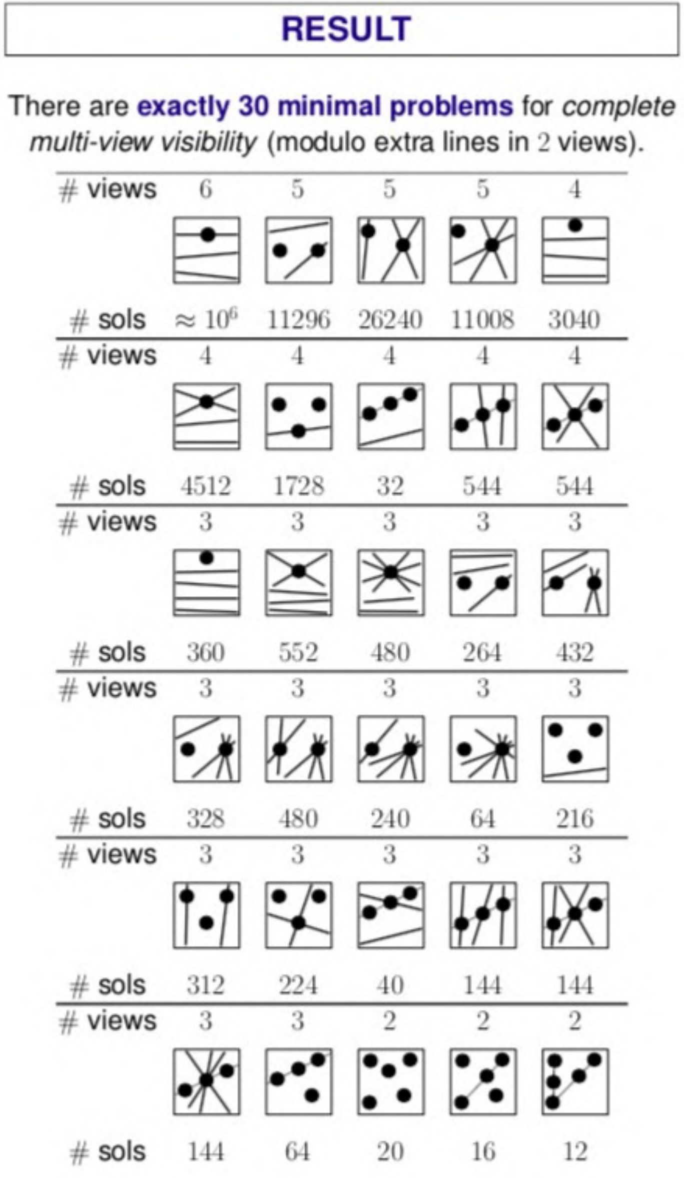


Our Tools: Nonlinear Algebra

- Algebraic geometry
  for proof of classification
- Gröbner bases
  symbolic computation of #sols
  for 2 & 3 views
- Homotopy continuation & monodromy
  numerical computation of #sols
  for 4, 5 & 6 views







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proof idea

The **joint camera map** is

$$\Phi : \mathcal{C}_m \times \mathcal{X} \longrightarrow \mathcal{Y}^m,$$

where

- $\mathcal{C}_m$ is the variety of m cameras,
- $\mathcal{X}$ is the variety of 3D arrangements of p points & ℓ lines satisfying some prescribed incidences, and
- $\mathcal{Y}$ is the analog variety of 2D arrangements.

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- $\mathcal{Y}$ is the analog variety of 2D arrangements.

3D reconstruction = given $(\mathrm{pic}_1, \ldots, \mathrm{pic}_m) \in \mathcal{Y}^m$, compute the preimage $\Phi^{-1}(\mathrm{pic}_1, \ldots, \mathrm{pic}_m)$

proof idea

Fact 1 from algebraic geometry:

If a 3D reconstruction problem is minimal, then its joint camera map

$$\Phi : \mathcal{C}_m \times \mathcal{X} \longrightarrow \mathcal{Y}^m,$$

satisfies $\dim \mathcal{C}_m + \dim \mathcal{X} = m \dim \mathcal{Y}$.

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There are exactly 39 reconstruction problems satisfying this dimension count:

m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p, l, \mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306^*	26240^*	11008^*	3040^*	4524^*			1728^*	32^*	544^*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544^*	360	552	480			264	432	328	480	240	64	216

m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{\mathrm{f}}p^{\mathrm{d}}l^{\mathrm{f}}l^{\mathrm{a}}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p, l, \mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

proof idea

Fact 2 from algebraic geometry:
A reconstruction problem with $\dim \mathcal{C}_m + \dim \mathcal{X} = m \dim \mathcal{Y}$ is minimal if and only if its joint camera map $\Phi : \mathcal{C}_m \times \mathcal{X} \longrightarrow \mathcal{Y}^m$ is **dominant**.

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Definition
A map $\varphi : A \to B$ is **surjective** if for every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

Definition
A map $\varphi : A \to B$ is **dominant** if for almost every $b \in B$ there is an $a \in A$ such that $\varphi(a) = b$.

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Fact 3 from algebraic geometry:
The joint camera map $\Phi : \mathcal{C}_m \times \mathcal{X} \longrightarrow \mathcal{Y}^m$ is dominant if and only if its derivative at a generic point in the domain is surjective.

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The joint camera map $\Phi : \mathcal{C}_m \times \mathcal{X} \longrightarrow \mathcal{Y}^m$ is dominant if and only if its derivative at a generic point in the domain is surjective.

Can check this computationally! It is only linear algebra!

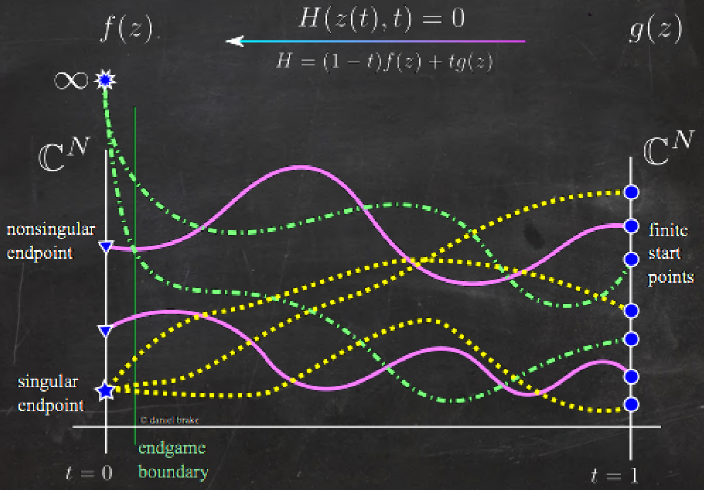
m views	6	6	6	5	5	5	4	4	4	4	4	4	4
$p^{f}p^{d}l^{f}l^{a}_{\alpha}$	1021_1	1013_3	1005_5	2011_1	2003_2	2003_3	1030_0	1022_2	1014_4	1006_6	3001_1	2110_0	2102_1
$(p,l,\mathcal{I})$													
Minimal	Y	N	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y
Degree	$> 450k^*$			11306*	26240*	11008*	3040*	4524*			1728*	32*	544*

m views	4	3	3	3	3	3	3	3	3	3	3	3	3
$p^{f}p^{d}l^{f}l^{a}_{\alpha}$	2102_2	1040_0	1032_2	1024_4	1016_6	1008_8	2021_1	2013_2	2013_3	2005_3	2005_4	2005_5	3010_0
$(p,l,\mathcal{I})$													
Minimal	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y
Degree	544*	360	552	480			264	432	328	480	240	64	216

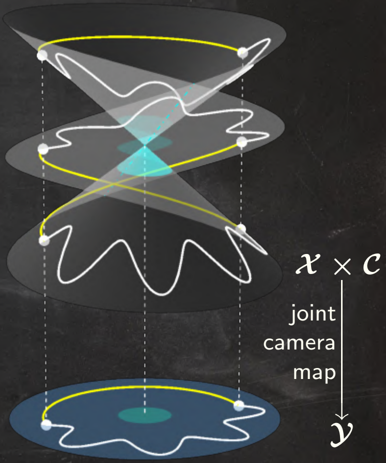
m views	3	3	3	3	3	3	3	3	2	2	2	2	2
$p^{f}p^{d}l^{f}l^{a}_{\alpha}$	3002_1	3002_2	2111_1	2103_1	2103_2	2103_3	3100_0	2201_1	5000_2	4100_3	3200_3	3200_4	2300_5
$(p,l,\mathcal{I})$													
Minimal	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	N
Degree	312	224	40	144	144	144	64		20	16	12		

Now that we have a complete list of minimal problems, we can find their solutions symbolically with Gröbner bases or numerically . . .

homotopy continuation

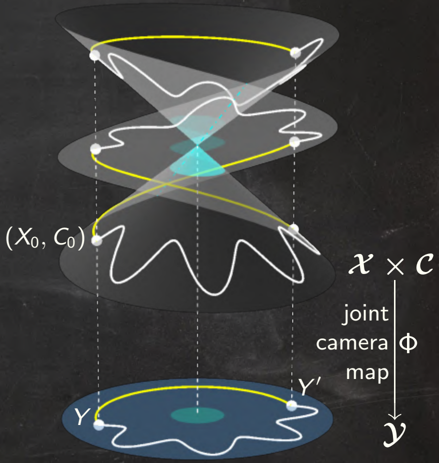


Monodromy



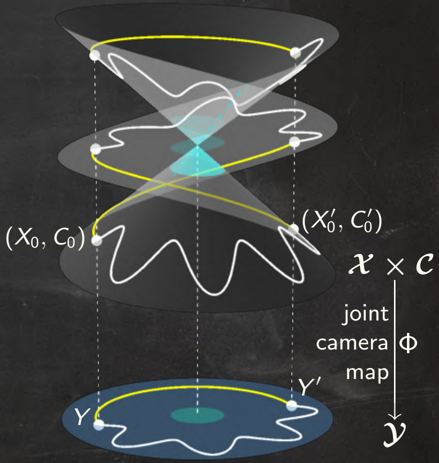
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$



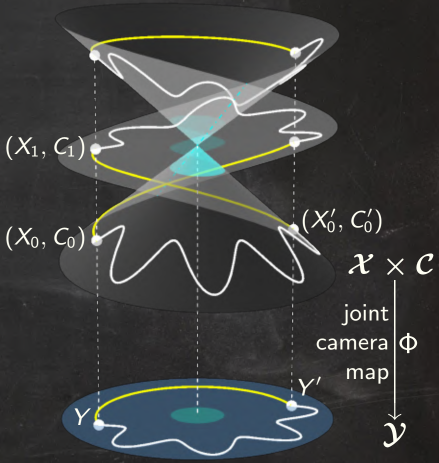
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**



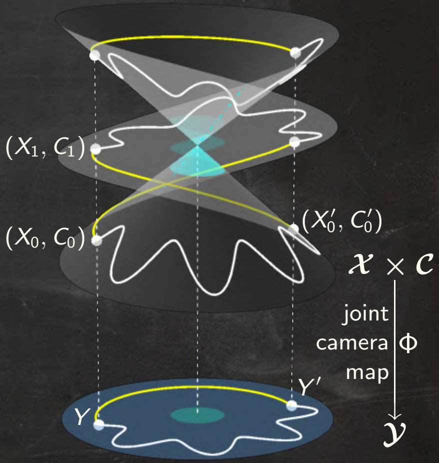
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**
- Along a random path from Y' to Y track the solution (X_0', C_0') for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**



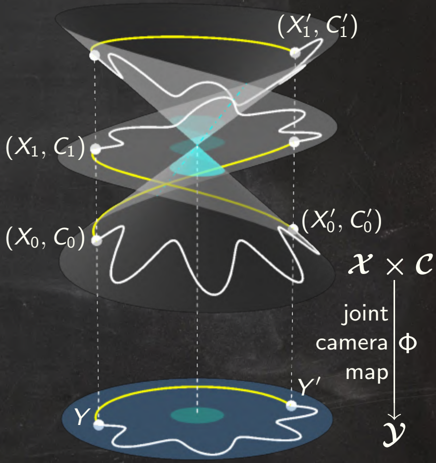
Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**
- Along a random path from Y' to Y track the solution (X_0', C_0') for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- Keep on circulating between Y and Y' until no more solutions for Y are found



Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X'_0, C'_0) for Y' via **homotopy continuation**
- Along a random path from Y' to Y track the solution (X'_0, C'_0) for Y' to a solution (X_1, C_1) for Y via **homotopy continuation**
- Keep on circulating between Y and Y' until no more solutions for Y are found



Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
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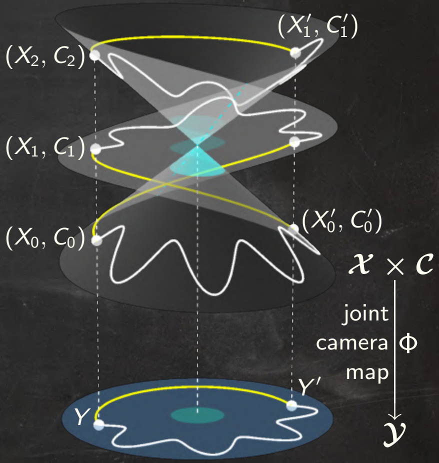
- Keep on circulating between Y and Y' until no more solutions for Y are found



Monodromy

- Pick random $(X_0, C_0) \in \mathcal{X} \times \mathcal{C}$
- Set $Y = \Phi(X_0, C_0)$
- Pick $Y' \in \mathcal{Y}$
- Along a random path from Y to Y' track the solution (X_0, C_0) for Y to a solution (X_0', C_0') for Y' via **homotopy continuation**
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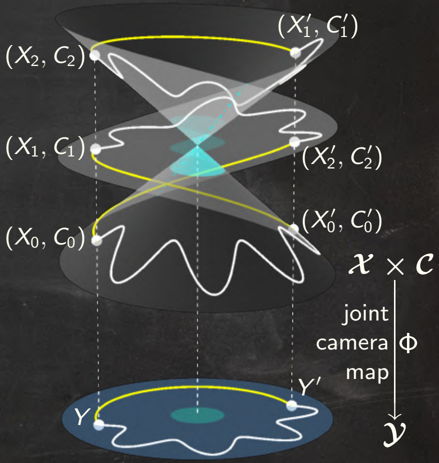
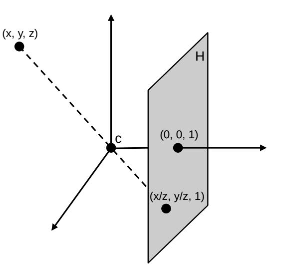
- Keep on circulating between Y and Y' until no more solutions for Y are found



What is a camera?

The shown classification of minimal problems was for **calibrated pinhole cameras**. This camera model is a linear projection and assumes the internal camera parameters to be known (and normalized).



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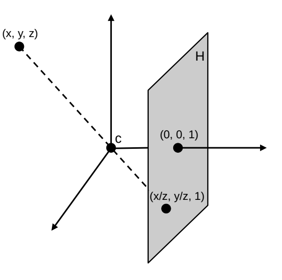


The unknown camera parameters are its center position and its orientation. They are modeled as a 3×4 matrix

$$[R \mid t], \qquad \text{where } R \in \mathrm{SO}(3), t \in \mathbb{R}^3,$$

that takes a picture of a point $X \in \mathbb{R}^3$ via $[R \mid t] \cdot \left[\begin{smallmatrix} X \\ 1 \end{smallmatrix}\right] = RX + t$.

What is a camera?

When the internal camera parameters are not known, the **uncalibrated pinhole camera** is an arbitrary full-rank 3×4 matrix.

What is a camera?

When the internal camera parameters are not known, the **uncalibrated pinhole camera** is an arbitrary full-rank 3×4 matrix.

Classifying all their minimal problems turns out to be more complicated . . .

Uncalibrated minimal problems

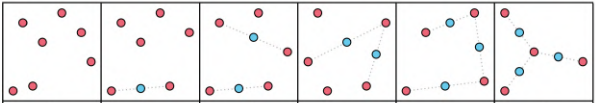
Theorem (K. Kiehn, A. Ahlbäck, K. Kohn; ICCV 2025):
For uncalibrated cameras, all minimal problems involving points and lines are:

a) 2 cameras viewing one of the point-line arrangements in Table 1, plus arbitrarily many additional lines;

b) at least 2 cameras observing one of the 2 right-most point-line arrangements in Table 1;

c) one of the 285 PLPs in SM Section E (with 3–9 views).

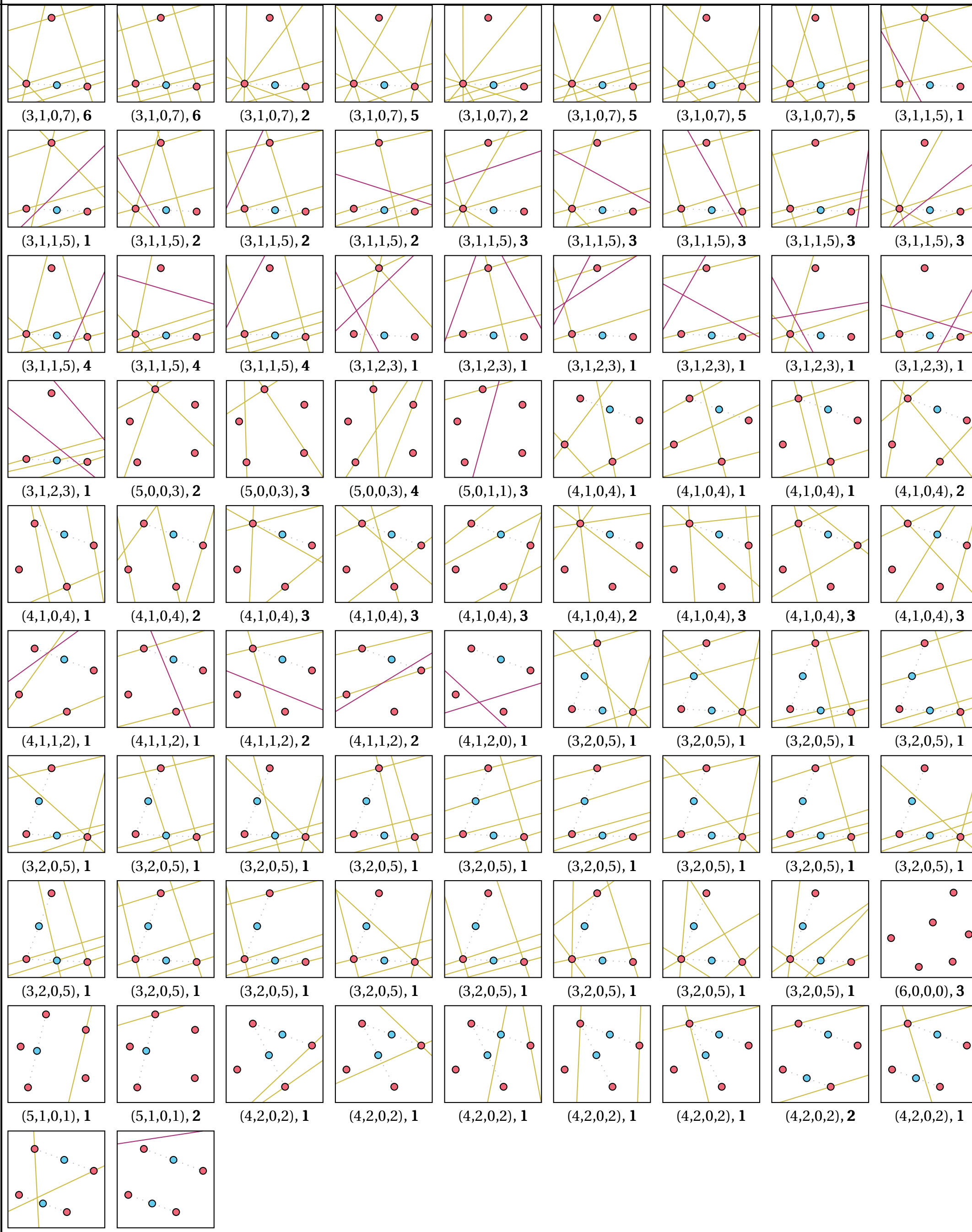
Their degrees are given in Table 1 and SM Section E.



3	2	2	1	1	1

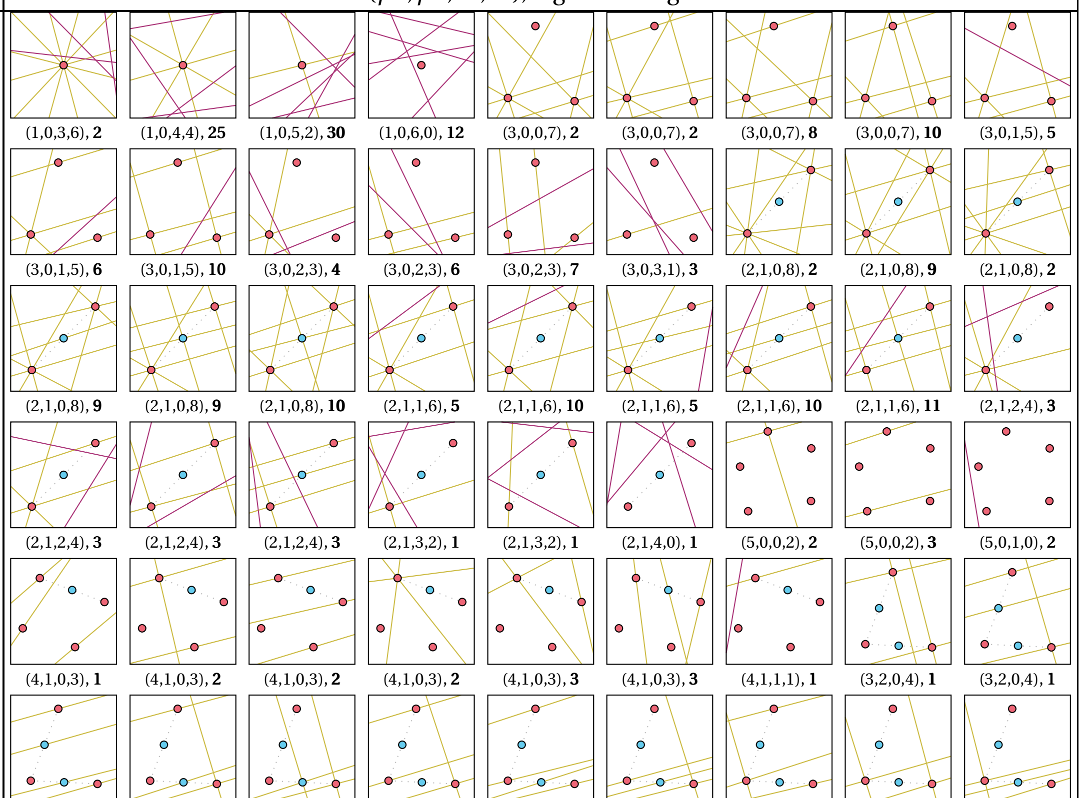
m	(p^f, p^d, l^f, l^a), algebraic degree								
3	(0,0,9,0), **36**	(1,0,4,7), **6**	(1,0,5,5), **23**	(1,0,6,3), **23**	(1,0,7,1), **15**	(2,0,0,12), **4**	(2,0,1,10), **6**	(2,0,1,10), **16**	(2,0,2,8), **4**
	(2,0,2,8), **12**	(2,0,2,8), **16**	(2,0,3,6), **2**	(2,0,3,6), **9**	(2,0,3,6), **15**	(2,0,3,6), **17**	(2,0,4,4), **9**	(2,0,4,4), **12**	(2,0,4,4), **13**
	(2,0,5,2), **8**	(2,0,5,2), **9**	(2,0,6,0), **7**	(3,0,0,9), **4**	(3,0,0,9), **4**	(3,0,0,9), **4**	(3,0,0,9), **10**	(3,0,0,9), **10**	(3,0,0,9), **12**
	(3,0,1,7), **2**	(3,0,1,7), **7**	(3,0,1,7), **2**	(3,0,1,7), **7**	(3,0,1,7), **10**	(3,0,1,7), **11**	(3,0,2,5), **2**	(3,0,2,5), **5**	(3,0,2,5), **7**
	(3,0,2,5), **8**	(3,0,2,5), **9**	(3,0,3,3), **6**	(3,0,3,3), **6**	(3,0,3,3), **6**	(3,0,4,1), **3**	(2,1,0,10), **4**	(2,1,0,10), **4**	(2,1,0,10), **4**
	(2,1,0,10), **4**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,1,8), **2**	(2,1,1,8), **7**	(2,1,1,8), **10**	(2,1,1,8), **2**
	(2,1,1,8), **7**	(2,1,1,8), **10**	(2,1,1,8), **10**	(2,1,1,8), **11**	(2,1,2,6), **2**	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,2,6), **5**
	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,4,2), **1**	(2,1,4,2), **1**	(2,1,5,0), **1**
	(4,0,0,6), **2**	(4,0,0,6), **5**	(4,0,0,6), **2**	(4,0,0,6), **5**	(4,0,0,6), **6**	(4,0,0,6), **5**	(4,0,0,6), **7**	(4,0,1,4), **3**	(4,0,1,4), **5**
	(4,0,1,4), **5**	(4,0,1,4), **6**	(4,0,2,2), **3**	(4,0,2,2), **4**	(4,0,3,0), **3**	(3,1,0,7), **2**	(3,1,0,7), **2**	(3,1,0,7), **2**	(3,1,0,7), **2**
	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **6**	(3,1,0,7), **5**

Table 7: Minimal problems with their associated degree.



m	(p^f, p^d, l^f, l^a), algebraic degree								
3	(3,1,0,7), **6**	(3,1,0,7), **6**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,1,5), **1**
	(3,1,1,5), **1**	(3,1,1,5), **2**	(3,1,1,5), **2**	(3,1,1,5), **2**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**
	(3,1,1,5), **4**	(3,1,1,5), **4**	(3,1,1,5), **4**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**
	(3,1,2,3), **1**	(5,0,0,3), **2**	(5,0,0,3), **3**	(5,0,0,3), **4**	(5,0,1,1), **3**	(4,1,0,4), **1**	(4,1,0,4), **1**	(4,1,0,4), **1**	(4,1,0,4), **2**
	(4,1,0,4), **1**	(4,1,0,4), **2**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **2**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **3**
	(4,1,1,2), **1**	(4,1,1,2), **1**	(4,1,1,2), **2**	(4,1,1,2), **2**	(4,1,2,0), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**
	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**
	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(6,0,0,0), **3**
	(5,1,0,1), **1**	(5,1,0,1), **2**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **2**	(4,2,0,2), **1**
	(4,2,0,2), **1**	(4,2,1,0), **1**							

Table 8: Minimal problems with their associated degree.



m	(p^f, p^d, l^f, l^a), algebraic degree								
4	(1,0,3,6), **2**	(1,0,4,4), **25**	(1,0,5,2), **30**	(1,0,6,0), **12**	(3,0,0,7), **2**	(3,0,0,7), **2**	(3,0,0,7), **8**	(3,0,0,7), **10**	(3,0,1,5), **5**
	(3,0,1,5), **6**	(3,0,1,5), **10**	(3,0,2,3), **4**	(3,0,2,3), **6**	(3,0,2,3), **7**	(3,0,3,1), **3**	(2,1,0,8), **2**	(2,1,0,8), **9**	(2,1,0,8), **2**
	(2,1,0,8), **9**	(2,1,0,8), **9**	(2,1,0,8), **10**	(2,1,1,6), **5**	(2,1,1,6), **10**	(2,1,1,6), **5**	(2,1,1,6), **10**	(2,1,1,6), **11**	(2,1,2,4), **3**
	(2,1,2,4), **3**	(2,1,2,4), **3**	(2,1,2,4), **3**	(2,1,3,2), **1**	(2,1,3,2), **1**	(2,1,4,0), **1**	(5,0,0,2), **2**	(5,0,0,2), **3**	(5,0,1,0), **2**
	(4,1,0,3), **1**	(4,1,0,3), **2**	(4,1,0,3), **2**	(4,1,0,3), **2**	(4,1,0,3), **3**	(4,1,0,3), **3**	(4,1,1,1), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**
	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**

Table 9: Minimal problems with their associated degree.

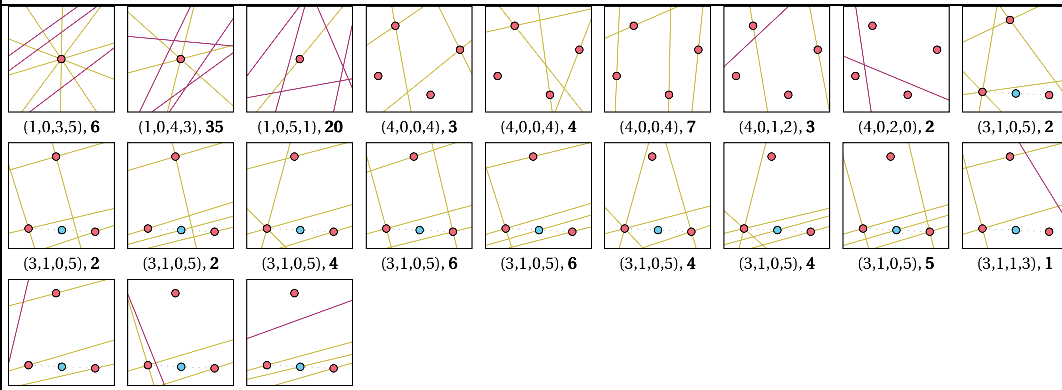
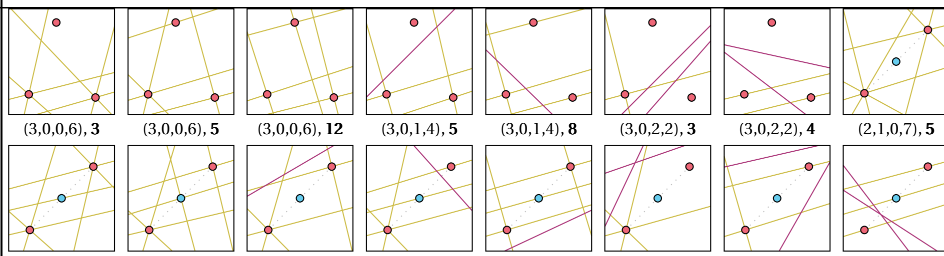
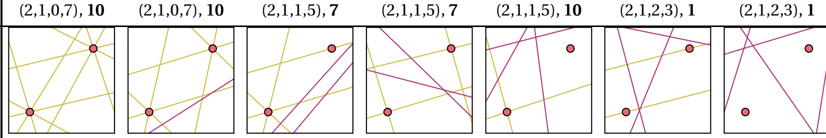
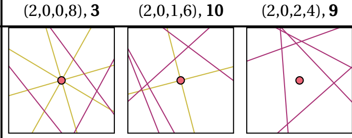
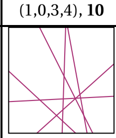
m	(p^f, p^d, l^f, l^a), algebraic degree
5	 (1,0,3,5), **6**; (1,0,4,3), **35**; (1,0,5,1), **20**; (4,0,0,4), **3**; (4,0,0,4), **4**; (4,0,0,4), **7**; (4,0,1,2), **3**; (4,0,2,0), **2**; (3,1,0,5), **2**; (3,1,0,5), **2**; (3,1,0,5), **2**; (3,1,0,5), **4**; (3,1,0,5), **6**; (3,1,0,5), **6**; (3,1,0,5), **4**; (3,1,0,5), **4**; (3,1,0,5), **5**; (3,1,1,3), **1**; (3,1,1,3), **1**; (3,1,1,3), **2**; (3,1,1,3), **2**
6	 (3,0,0,6), **3**; (3,0,0,6), **5**; (3,0,0,6), **12**; (3,0,1,4), **5**; (3,0,1,4), **8**; (3,0,2,2), **3**; (3,0,2,2), **4**; (2,1,0,7), **5**; (2,1,0,7), **5**; (2,1,0,7), **10**; (2,1,0,7), **10**; (2,1,1,5), **7**; (2,1,1,5), **7**; (2,1,1,5), **10**; (2,1,2,3), **1**; (2,1,2,3), **1**; (2,1,2,3), **1**
7	 (2,0,0,8), **3**; (2,0,1,6), **10**; (2,0,2,4), **9**; (2,0,2,4), **20**; (2,0,3,2), **6**; (2,0,3,2), **9**; (2,0,4,0), **3**
8	 (1,0,3,4), **10**; (1,0,4,2), **38**; (1,0,5,0), **8**
9	 (0,0,6,0), **114**

Table 10: Minimal problems with their associated degree.

important challenge: algebra-geometry foundations of

rolling-shutter cameras that are the vast majority of today's cameras:
take pictures by scanning across the scene, capturing the image row by row

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rolling-shutter cameras that are the vast majority of today's cameras:
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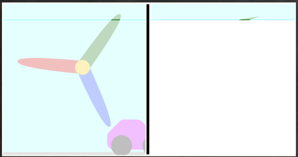
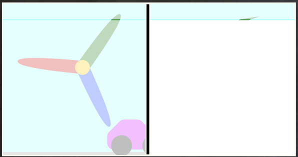
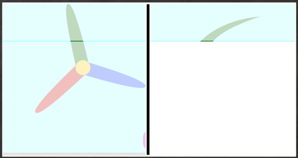


important challenge: algebra-geometry foundations of

rolling-shutter cameras that are the vast majority of today's cameras: take pictures by scanning across the scene, capturing the image row by row





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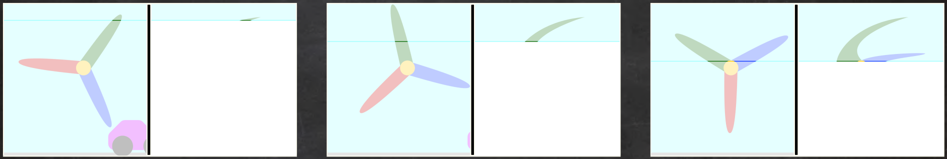

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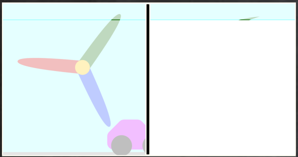

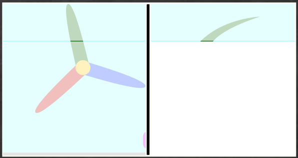

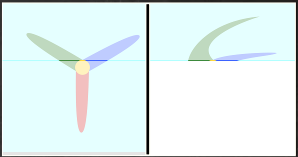

Algebraically:

- The image of a line is typically a higher-degree curve.

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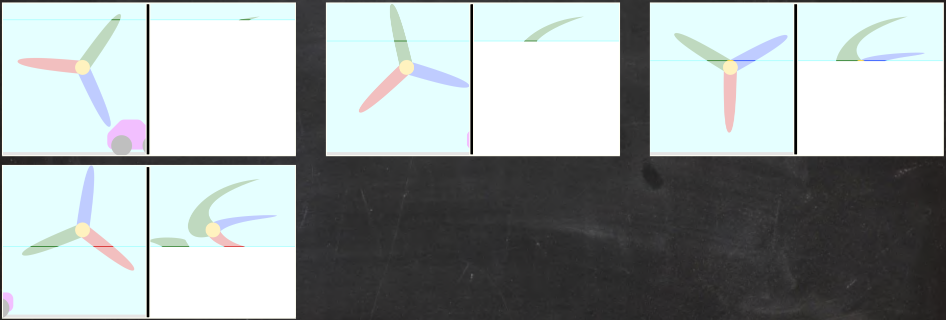

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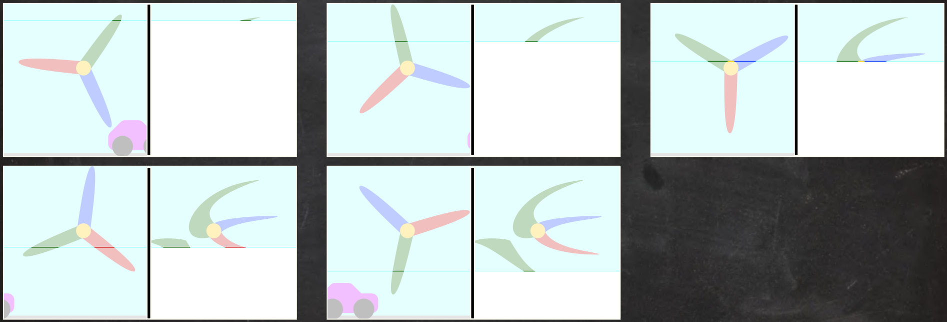

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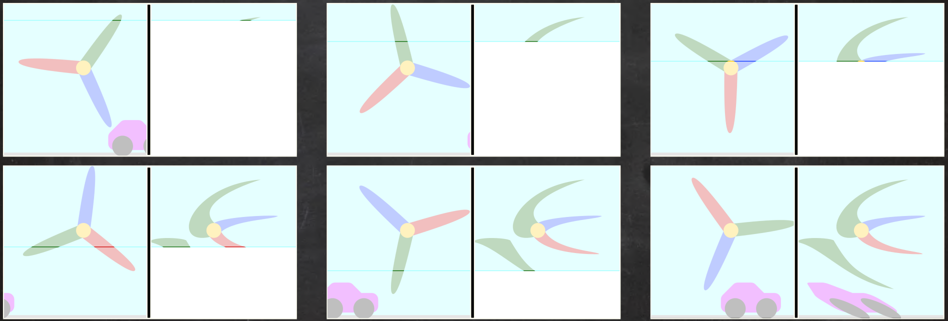

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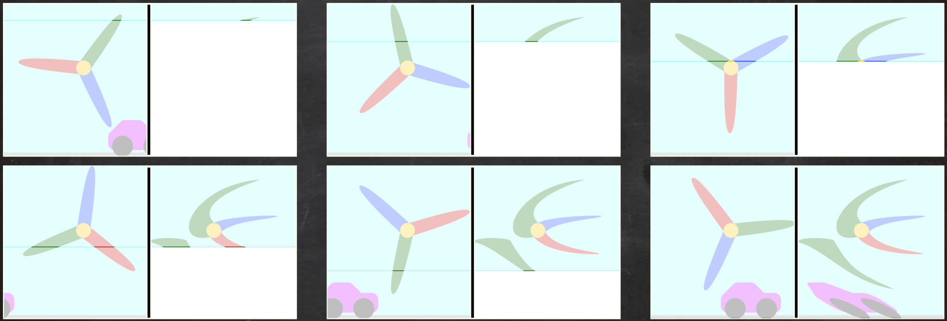

(by Cmglee @ Wikipedia
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changes: added black separating line)

Algebraically:

- The image of a line is typically a higher-degree curve.
- A 3D point can appear more than once in the image.