The Adjoint of a Polytope

joint works with Kristian Ranestad (Universitetet i Oslo) /
Boris Shapiro (Stockholms universitet) & Bernd Sturmfels (MPI MiS Leipzig, UC Berkeley)

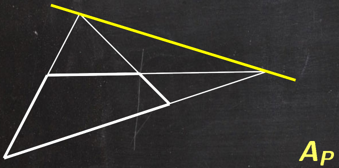
April 13, 2019

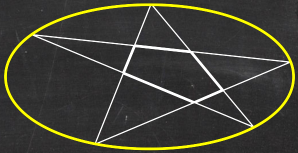
The Adjoint of a Polygon

Wachspress (1975)

Definition

The **adjoint A_P** of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.





$(\deg A_P = |V(P)| - 3)$

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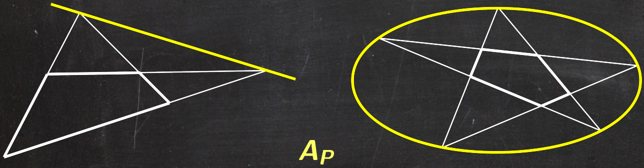

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Generalization to higher-dimensional polytopes?

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

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Definition

$$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

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$\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$. *So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

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(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ dual polytope of P)

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Geometric definition using a vanishing condition à la Wachspress?

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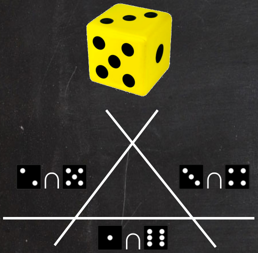

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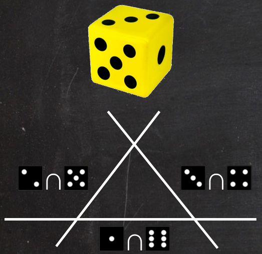

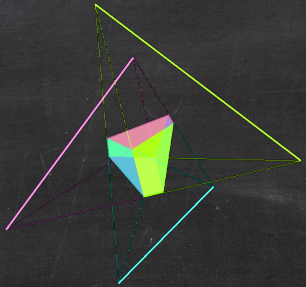

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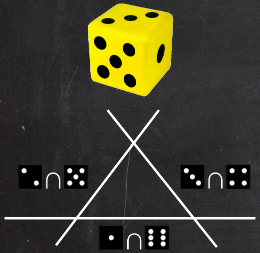

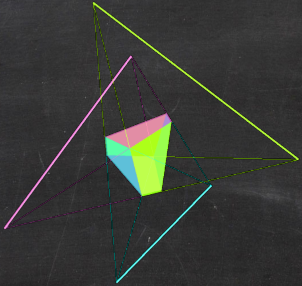

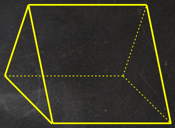

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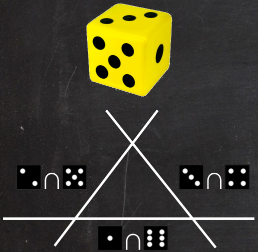

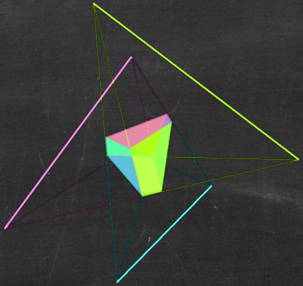

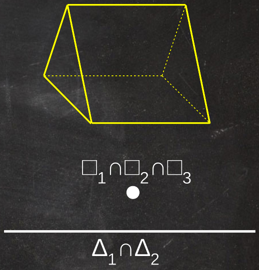

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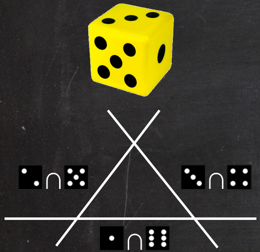

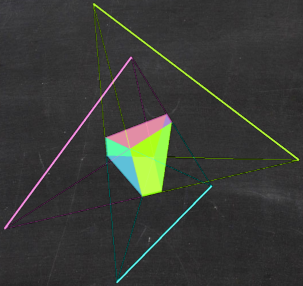

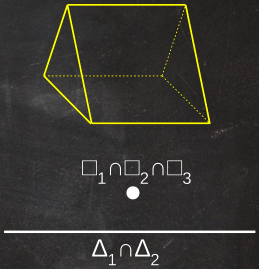

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

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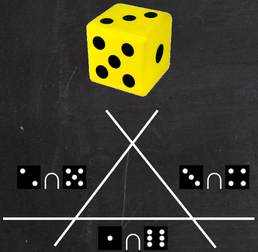

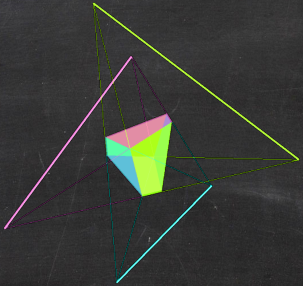

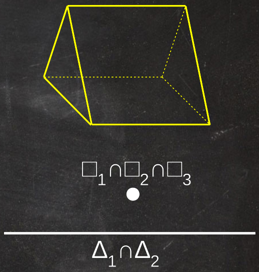

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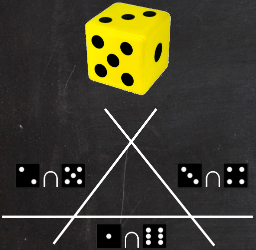

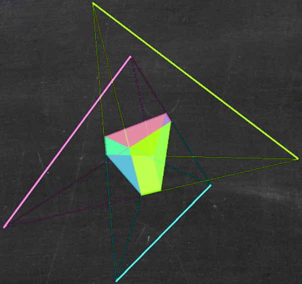

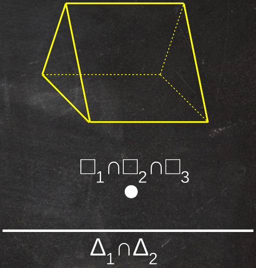

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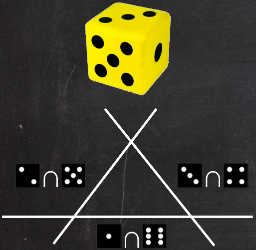

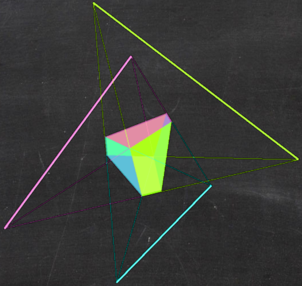

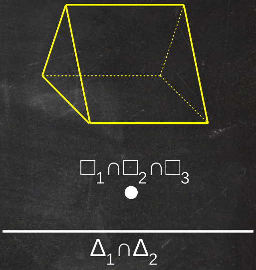

adjoint plane

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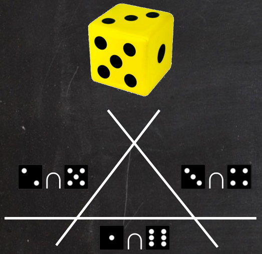
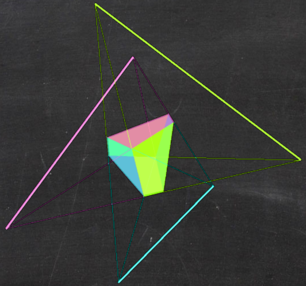
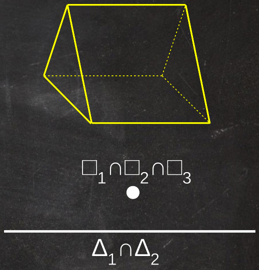




adjoint quadric surface



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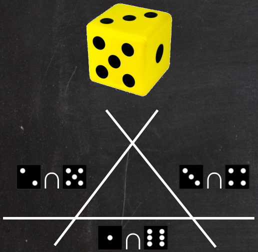
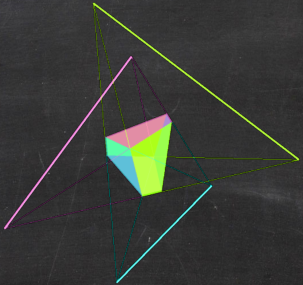


adjoint double plane



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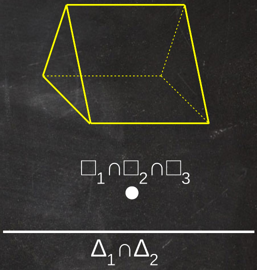

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Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

Application 1: Barycentric Coordinates

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Definition
The **Wachspress coordinates** of P are

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}.$$

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

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$$m_{\mathcal{I}}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \dots w_n^{i_n} d\mu_P \quad \text{for } \mathcal{I} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$$

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Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}}\, m_{\mathcal{I}}(P)\, t^{\mathcal{I}} = \frac{\mathrm{adj_P(t)}}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

$$\textit{where } c_{\mathcal{I}} := \binom{i_1 + i_2 + \dots + i_n + n}{i_1, i_2, \dots, i_n, n}.$$

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P

Idea:

$$P \rightsquigarrow \mathcal{H}_P$$

hypersurface
of degree d

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Idea:

$$P \rightsquigarrow \mathcal{H}_P \rightsquigarrow D$$

hypersurface of degree d

polytopal hypersurface: hypersurface of degree d, multiplicity c along $\mathcal{R}_P^c$, smooth outside of $\mathcal{R}_P$

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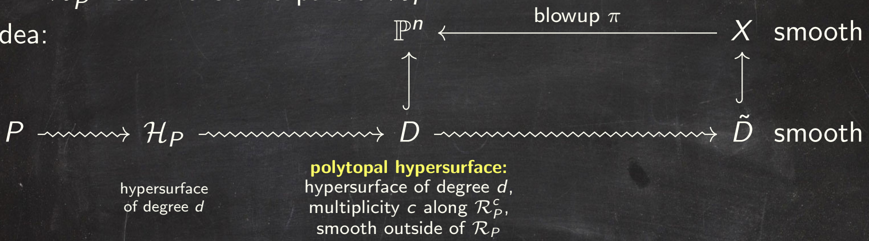

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Idea:

$$\begin{array}{ccccccc} & & & & \mathbb{P}^n & \xleftarrow{\text{blowup } \pi} & X \quad \text{smooth} \\ & & & & \uparrow & & \uparrow \\ P & \rightsquigarrow & \mathcal{H}_P & \rightsquigarrow & D & \rightsquigarrow & \tilde{D} \quad \text{smooth} \end{array}$$

hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
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Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

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- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

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$\mathcal{H}_P$: hypersurface of degree d

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Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

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$D \hookrightarrow \mathbb{P}^n$, $\tilde{D} \hookrightarrow X$

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When can we find a polytopal hypersurface D?

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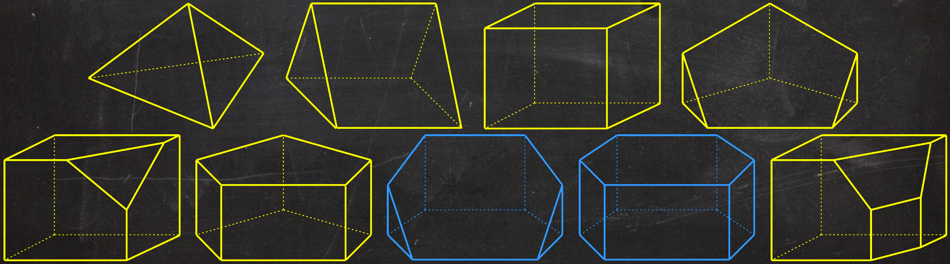
Theorem (K., Ranestad)

Let $\mathcal{C}$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let P be a general polytope of type $\mathcal{C}$. There is a polytopal surface D iff $\mathcal{C}$ is one of:



In that case, the general D is either an elliptic surface or a K3-surface.