The Adjoint of a Polytope

joint works with Kristian Ranestad (Universitetet i Oslo) /
Boris Shapiro (Stockholms universitet) & Bernd Sturmfels (MPI MiS Leipzig, UC Berkeley)

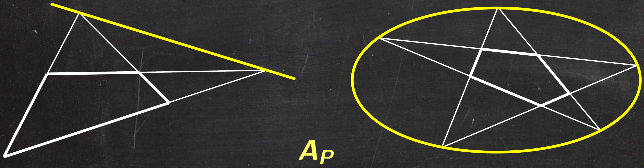
April 14, 2019

The Adjoint of a Polygon

Wachspress (1975)

Definition

The **adjoint** $\boldsymbol{A_P}$ of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.



$(\deg A_P = |V(P)| - 3)$

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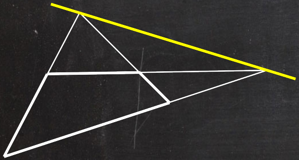
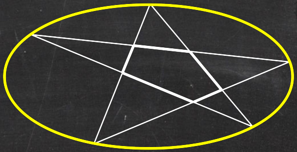


A_P



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Generalization to higher-dimensional polytopes?

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
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Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

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Theorem (Warren)

$\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$*. So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

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II *If* P *is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.

(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ *dual polytope of* P*)*

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Geometric definition using a vanishing condition à la Wachspress?

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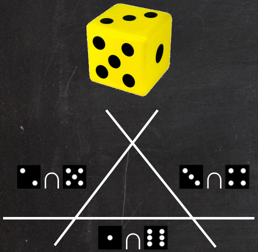

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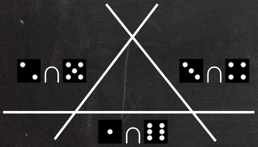

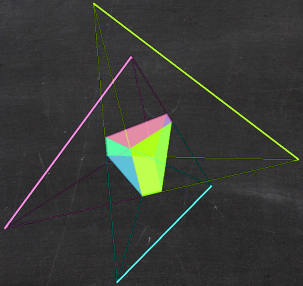

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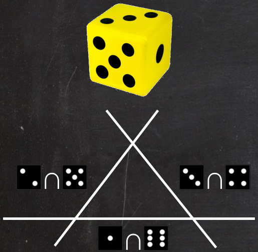

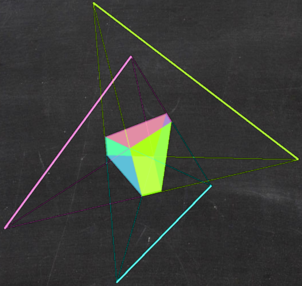

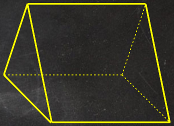

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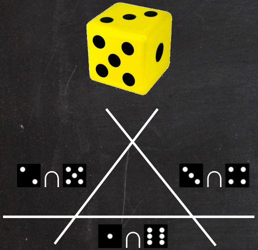

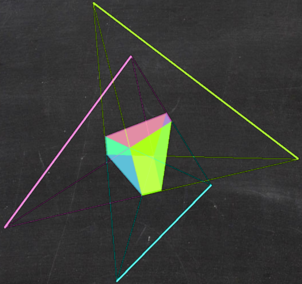

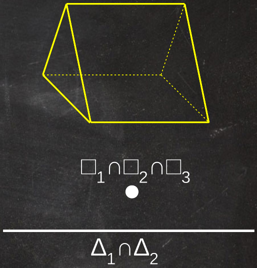

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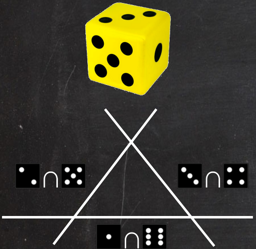

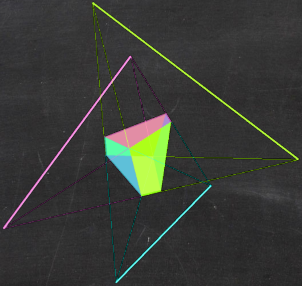

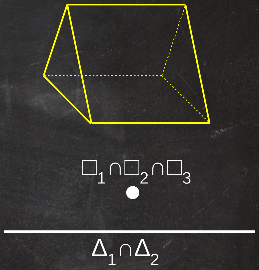

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

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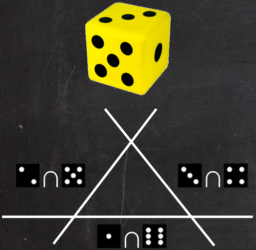

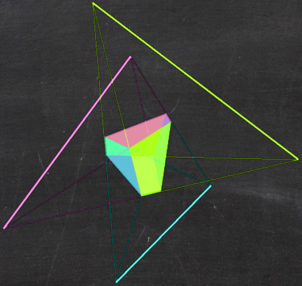

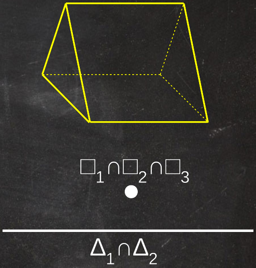

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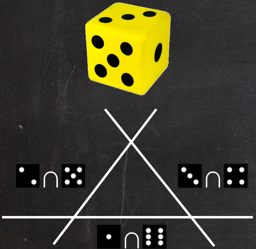

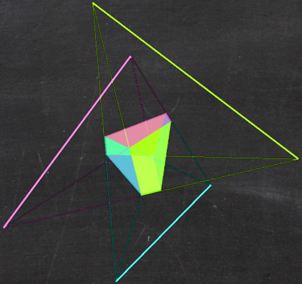

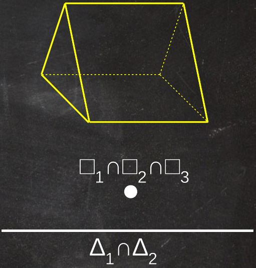

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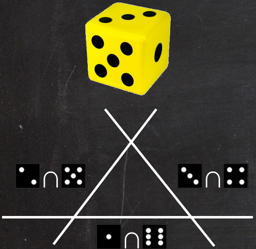

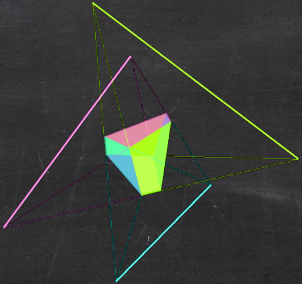

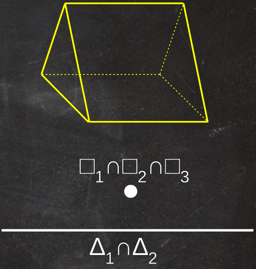

adjoint plane

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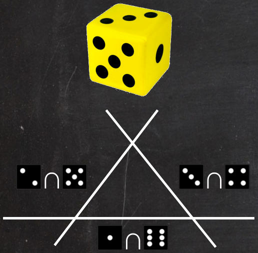
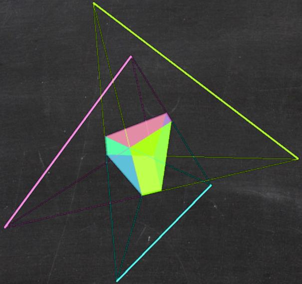
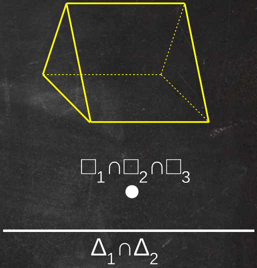




adjoint quadric surface



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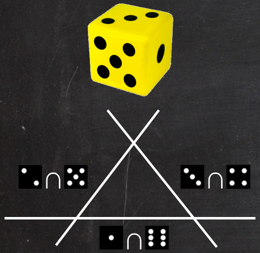
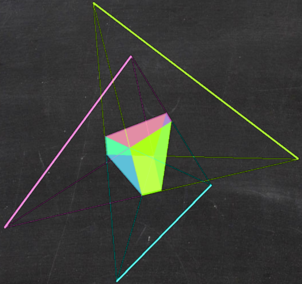


adjoint double plane



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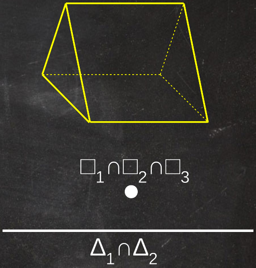

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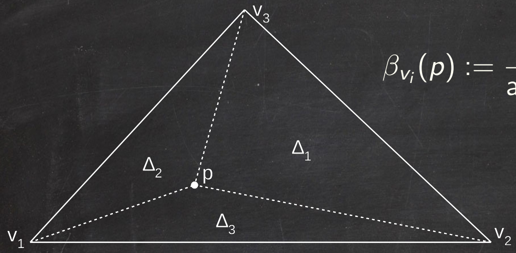
Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

Application 1: Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

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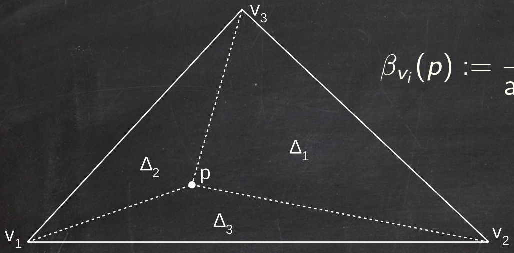

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Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

(i) $\forall u \in V(P) : \beta_u(p) > 0$,

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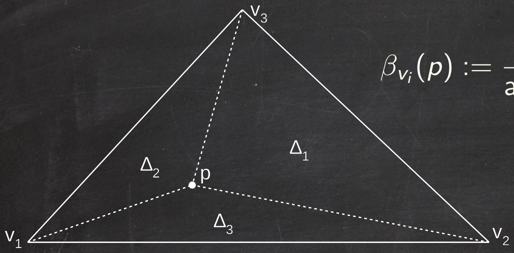

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Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!

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For other GBCs and applications of GBCs (e.g., mesh parameterizations in geometric modelling, deformations in computer graphics, or polyhedral FEM):
[Floater: Generalized barycentric coordinates and applications, Acta Numerica 24 (2015)]

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

- P: convex polytope in $\mathbb{R}^n$
- μ_P: uniform probability distribution on P

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

- P: convex polytope in $\mathbb{R}^n$
- μ_P: uniform probability distribution on P
- moments

$$m_{\mathcal{I}}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \dots w_n^{i_n} \, d\mu_P \quad \text{for } \mathcal{I} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$$

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Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}} \, m_{\mathcal{I}}(P) \, t^{\mathcal{I}} = \frac{\mathrm{adj_P(t)}}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

where $c_{\mathcal{I}} := \binom{i_1 + i_2 + \ldots + i_n + n}{i_1, i_2, \ldots, i_n, n}$.

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- $\mathcal{H}_\mathcal{P}$: simple hyperplane arrangement spanned by facets of P

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D: **polytopal hypersurface:** hypersurface of degree d, multiplicity c along $\mathcal{R}_P^c$, smooth outside of $\mathcal{R}_P$

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When can we find a polytopal hypersurface D?

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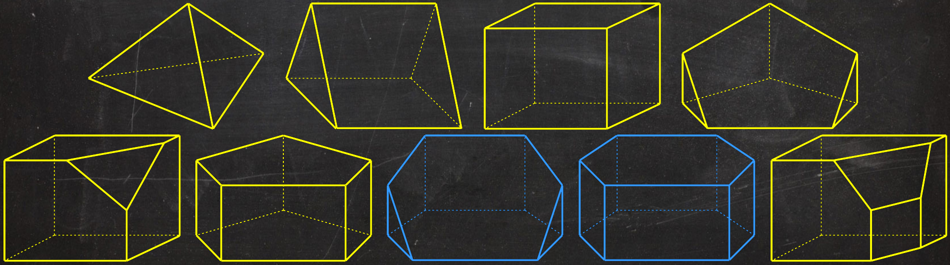
Theorem (K., Ranestad)

Let $\mathcal{C}$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let P be a general polytope of type $\mathcal{C}$. There is a polytopal surface D iff $\mathcal{C}$ is one of:



In that case, the general D is either an elliptic surface or a K3-surface.