The Adjoint of a Polytope

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The Adjoint of a Polygon  
Wachspress (1975)

Definition
The **adjoint** $A_P$ of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of $P$.

$A_P$  
(deg $A_P = |V(P)| - 3$)
The Adjoint of a Polygon
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$\deg A_P = |V(P)| - 3$

Generalization to higher-dimensional polytopes?
The Adjoint of a Polytope

Warren (1996)

- $P$: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of $P$
- $\tau(P)$: triangulation of $P$ using only the vertices of $P$

**Definition**

$\text{adj} \tau(P)(t) := \sum_{\sigma \in \tau(P)} \text{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t)$,

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \cdots - v_n t_n$.

**Theorem (Warren)**

$\text{adj} \tau(P)(t)$ is independent of the triangulation $\tau(P)$.

So $\text{adj} P := \text{adj} \tau(P)$.

II - If $P$ is a polygon, then $Z(\text{adj} P) = A_P^*$.

(Recall: $P^* = \{x \in \mathbb{R}^n | \forall v \in V(P): \ell_v(x) \geq 0\}$ dual polytope of $P$.)
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**Definition**

\[ \text{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \text{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t), \]

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$. 

**Theorem** (Warren)

I. $\text{adj}_{\tau(P)}(P)(t)$ is independent of the triangulation $\tau(P)$. So $\text{adj}(P) := \text{adj}_{\tau(P)}(P)$.

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Geometric definition using a vanishing condition à la Wachspress?
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Theorem (K., Ranestad)

If \( \mathcal{H}_P \) is simple (i.e. through any point in \( \mathbb{P}^n \) pass \( \leq n \) hyperplanes), there is a unique hypersurface \( A_P \) in \( \mathbb{P}^n \) of degree \( d - n - 1 \) passing through \( \mathcal{R}_P \). \( A_P \) is called the adjoint of \( P \).
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If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface $A_P$ in $\mathbb{P}^n$ of degree $d-n-1$ passing through $\mathcal{R}_P$. $A_P$ is called the adjoint of $P$. 
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**Theorem Diagrams**

- Adjoint double plane
- Adjoint quadric surface
- Adjoint plane
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**Proposition (K., Ranestad)**

Warren's adjoint polynomial $\text{adj}_P$ vanishes along $\mathcal{R}_P^*$. If $\mathcal{H}_P^*$ is simple, then $Z(\text{adj}_P) = A_P^*$. 

IV - X
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**Proposition (K., Ranestad)**

Warren's adjoint polynomial $\text{adj}_P$ vanishes along $\mathcal{R}_P^\ast$. If $\mathcal{H}_P^\ast$ is simple, then $Z(\text{adj}_P) = A_P^\ast$. 
Application 1: Barycentric Coordinates

\[ \beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)} \]

for \( i = 1, 2, 3 \)
Application 1: Barycentric Coordinates

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**Definition**

Let \( P \) be a convex polytope in \( \mathbb{R}^n \). A set of functions \( \{ \beta_u : P^\circ \rightarrow \mathbb{R} \mid u \in V(P) \} \) is called generalized barycentric coordinates for \( P \) if, for all \( p \in P^\circ \),

(i) \( \forall u \in V(P) : \beta_u(p) > 0 \),

(ii) \( \sum_{u \in V(P)} \beta_u(p) = 1 \), and

(iii) \( \sum_{u \in V(P)} \beta_u(p) u = p \).
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Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!
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Proposition (Warren)

The Wachspress coordinates

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\beta_u(t) := \frac{\text{adj} F_u(t)}{\prod_{F \in \mathcal{F}(P)} (u \in F) \ell_F(t)}
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for \( u \in V(P) \)

are generalized barycentric coordinates for \( P \).

For other GBCs and applications of GBCs (e.g., mesh parameterizations in geometric modelling, deformations in computer graphics, or polyhedral FEM):

Application 1: Barycentric Coordinates

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\[ V(P) \overset{1:1}{\longleftrightarrow} \mathcal{F}(P^*) \]

\[ v \rightarrow F_v \]

Proposition (Warren)
The Wachspress coordinates $\beta_u(t) := \text{adj} F_u(t) \cdot \prod_{F \in \mathcal{F}(P)} u/\in F \ell v F(t)$ for $u \in V(P)$ are generalized barycentric coordinates for $P$.

For other GBCs and applications of GBCs (e.g., mesh parameterizations in geometric modelling, deformations in computer graphics, or polyhedral FEM): [Floater: Generalized barycentric coordinates and applications, Acta Numerica 24 (2015)]
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\[ \nu \mapsto F_\nu \]

\[ \mathcal{F}(P) \leftrightarrow_{1:1} V(P^*) \]
\[ F \mapsto v_F \]

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for $u \in V(P)$

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$$\beta_u(t) := \frac{\text{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P): u \notin F} \ell_{\nu_F}(t)}{\text{adj}_{P^*}(t)}$$

for $u \in V(P)$

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For other GBCs and applications of GBCs (e.g., mesh parameterizations in geometric modelling, deformations in computer graphics, or polyhedral FEM):

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

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- $\mu_P$: uniform probability distribution on $P$
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- $P$: convex polytope in $\mathbb{R}^n$
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- moments

$$m_{I}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \ldots w_n^{i_n} d\mu_P \quad \text{for } I = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$$
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- \( P \): convex polytope in \( \mathbb{R}^n \)
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m_\mathcal{I}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \ldots w_n^{i_n} d\mu_P \quad \text{for} \quad \mathcal{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_\geq^n
\]

Proposition (K., Shapiro, Sturmfels)

\[
\sum_{\mathcal{I} \in \mathbb{Z}_\geq^n} c_\mathcal{I} m_\mathcal{I}(P) t^\mathcal{I} = \frac{\text{adj}_P(t)}{\text{vol}(P) \prod_{v \in V(P)} \ell_v(t)},
\]

where \( c_\mathcal{I} := \binom{i_1+i_2+\ldots+i_n+n}{i_1,i_2,\ldots,i_n,n} \).
Application 3: Segre Classes of Monomial Schemes

Aluffi

- $V$: smooth variety
- $X_1, \ldots, X_n$: smooth hypersurfaces meeting with normal crossings in $V$

Theorem (Aluffi, (Harris, K., Ranestad))

The Segre class of $S_A$ in the Chow ring of $V$ is

$$n! X_1 \cdots X_n \text{adj}_N A \left( -X \right) \prod_{v \in V} (N_A) \ell_v \left( -X \right)$$

Example: $n = 2$

$A = \{ (2,6), (3,4), (4,3), (5,1), (7,0) \}$
Application 3: Segre Classes of Monomial Schemes

- **$V$:** smooth variety
- **$X_1, \ldots, X_n$:** smooth hypersurfaces meeting with normal crossings in $V$
- **$X^I$:** hypersurface obtained by taking $X_{i_j}$ with multiplicity $i_j$
  
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- $\mathcal{A} \subset \mathbb{Z}_{\geq 0}^n$ defines a **monomial subscheme**

$S_\mathcal{A} = \bigcap_{I \in \mathcal{A}} X^I$
Application 3: Segre Classes of Monomial Schemes

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$S_\mathcal{A} = \bigcap_{\mathcal{I} \in \mathcal{A}} X^\mathcal{I}$ and a Newton region $N_\mathcal{A} \subset \mathbb{R}^n_{\geq 0}$

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The Segre class of $S_\mathcal{A}$ in the Chow ring of $V$ is

$$\frac{n! \ X_1 \cdots X_n \ adj_{N_\mathcal{A}}(-X)}{\prod_{\nu \in V(N_\mathcal{A})} \ell_{\nu}(-X)}$$

**Example:** $n = 2$

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![Diagram](image)
Why “Adjoint”? 

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets 
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$

Idea:

$$P \xrightarrow{\text{hypersurface}} \mathcal{H}_P$$

hypersurface of degree $d$
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- $P$: polytope in $\mathbb{P}^n$ with $d$ facets
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P \xrightarrow{\text{hypersurface}} \mathcal{H}_P & \xrightarrow{\text{polytopal hypersurface}} D \\
& \text{hypersurface of degree } d, \\
& \text{multiplicity } c \text{ along } \mathcal{R}_P^c, \\
& \text{smooth outside of } \mathcal{R}_P
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\uparrow & & \uparrow \\
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\end{array}
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polytopal hypersurface: hypersurface of degree $d$, multiplicity $c$ along $\mathcal{R}^c_P$, smooth outside of $\mathcal{R}_P$
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\uparrow & & \uparrow \\
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\mathcal{H}_P & \xrightarrow{\text{hypersurface of degree } d, \text{ multiplicity } c \text{ along } \mathcal{R}_P^c, \text{ smooth outside of } \mathcal{R}_P} & \tilde{D} \\
\end{array} \]

Adjunction formula: 

\[ K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}} \]
Why “Adjoint”? 

- **P**: polytope in \( \mathbb{P}^n \) with \( d \) facets
- **\( \mathcal{H}_P \)**: simple hyperplane arrangement spanned by facets of \( P \)
- **\( \mathcal{R}_P^c \)**: codimension-\( c \) part of \( \mathcal{R}_P \)

Idea:

- Blowup \( \pi \)
- \( \mathbb{P}^n \leftarrow X \) smooth
- \( P \overset{\mathcal{H}_P}{\longrightarrow} D \) polytopal hypersurface: hypersurface of degree \( d \), multiplicity \( c \) along \( \mathcal{R}_P^c \), smooth outside of \( \mathcal{R}_P \)
- \( D \overset{\tilde{D}}{\longrightarrow} \tilde{D} \) smooth

**Adjunction formula**: \( K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}} \)

**Def.**: An **adjoint to \( \tilde{D} \) in \( X \)** is a hypersurface \( A \) in \( X \) s.t. \([A] = K_X + [\tilde{D}]\).
Why “Adjoint”?

- \( P \): polytope in \( \mathbb{P}^n \) with \( d \) facets
- \( \mathcal{H}_P \): simple hyperplane arrangement spanned by facets of \( P \)
- \( R^c_P \): codimension-\( c \) part of \( R_P \)

Idea:

\[
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{\text{blowup } \pi} & X \text{ smooth} \\
\uparrow & & \uparrow \\
P & \rightarrow & \mathcal{H}_P & \rightarrow & D & \rightarrow & \tilde{D} \text{ smooth}
\end{array}
\]

\( P \rightarrow \mathcal{H}_P \rightarrow D \rightarrow \tilde{D} \)

hypersurface of degree \( d \)

polytopal hypersurface: hypersurface of degree \( d \), multiplicity \( c \) along \( R^c_P \), smooth outside of \( R_P \)

Adjunction formula: \( K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}} \)

Def.: An \textbf{adjoint to } \tilde{D} \textbf{ in } X \text{ is a hypersurface } A \text{ in } X \text{ s.t. } [A] = K_X + [\tilde{D}] \).

Proposition (K., Ranestad)

\( \tilde{D} \) has a unique adjoint \( A \) in \( X \), and thus a unique canonical divisor: \( A \cap \tilde{D} \). Moreover, \( \pi(A) = A_P \).
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**Adjunction formula**: 
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Proposition (K., Ranestad)

Let $P$ be a general $d$-gon in $\mathbb{P}^2$. There is a polygonal curve $D$ iff $d \leq 6$. In that case, $D$ is an elliptic curve.
Polytopal Hypersurfaces

Proposition (K., Ranestad)
Let $P$ be a general $d$-gon in $\mathbb{P}^2$. There is a polygonal curve $D$ iff $d \leq 6$. In that case, $D$ is an elliptic curve.

Theorem (K., Ranestad)
Let $C$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let $P$ be a general polytope of type $C$. There is a polytopal surface $D$ iff $C$ is one of:

In that case, the general $D$ is either an elliptic surface or a K3-surface.
Thanks for your attention