The Adjoint of a Polytope

Kathlén Kohn (Universitetet i Oslo)

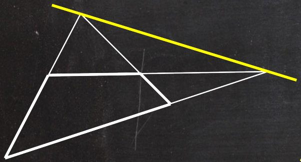
June 7, 2019

The Adjoint of a Polygon

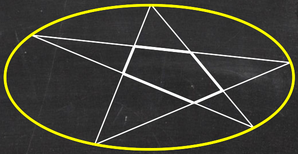
Wachspress (1975)

Definition

The **adjoint** $\boldsymbol{A_P}$ of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.



A_P



$(\deg A_P = |V(P)| - 3)$

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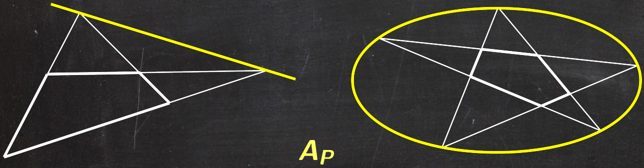

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Generalization to higher-dimensional polytopes?

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
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Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

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Theorem (Warren)

$\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$. *So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

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I $\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$*. So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.
II *If P is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.
(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ *dual polytope of P)*

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Geometric definition using a vanishing condition à la Wachspress?

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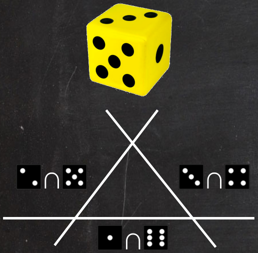

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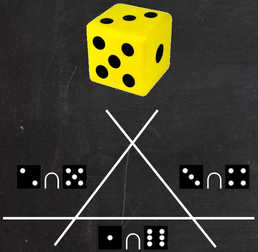

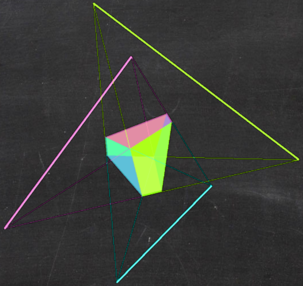

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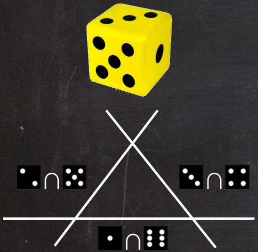

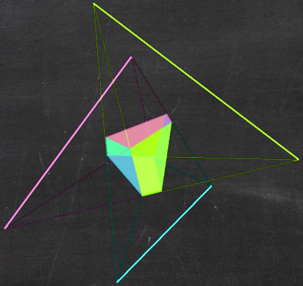

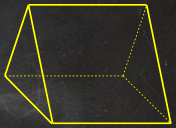

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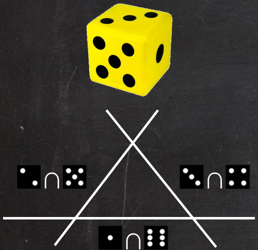

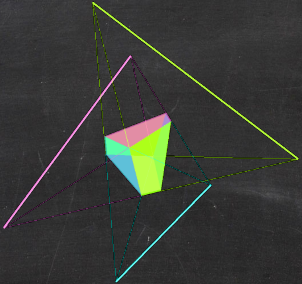

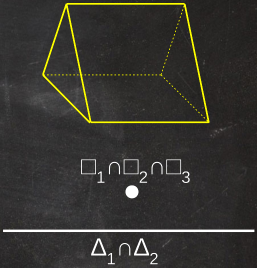

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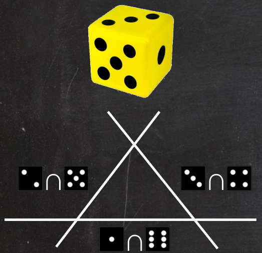

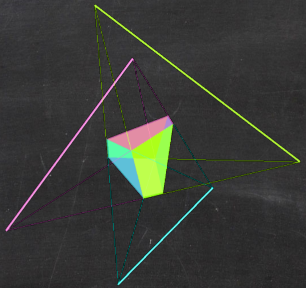

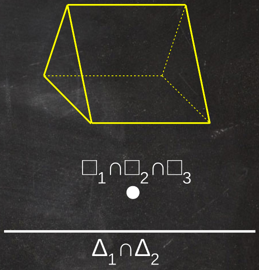

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

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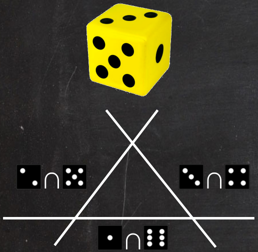

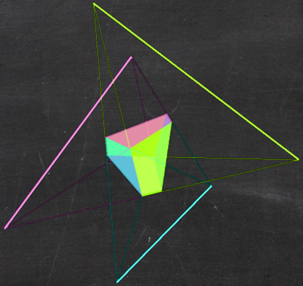

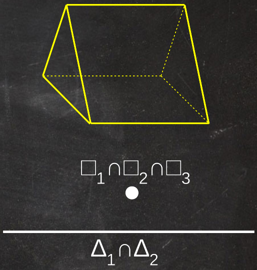

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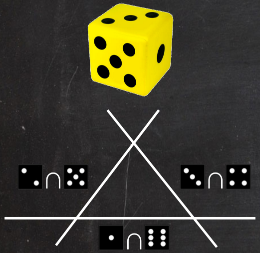

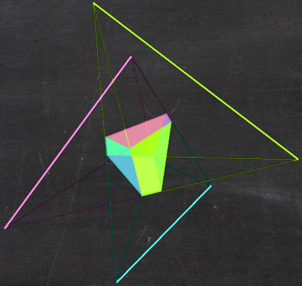

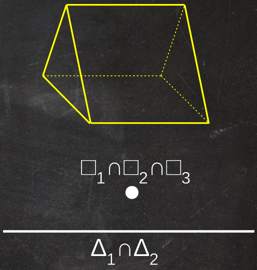

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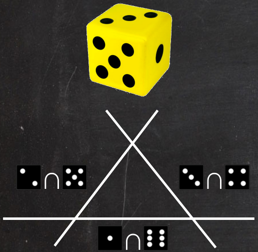

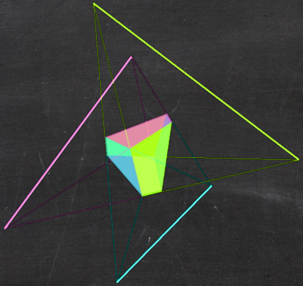

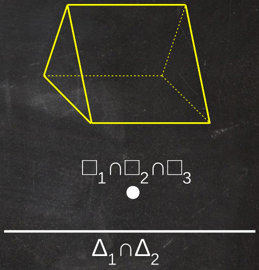

adjoint plane

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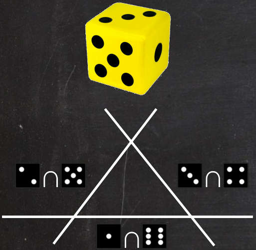
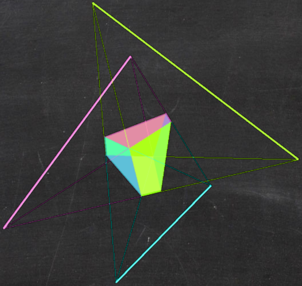
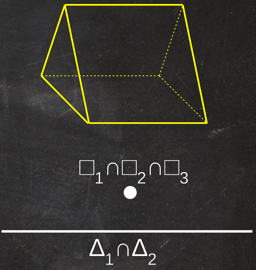




adjoint quadric surface



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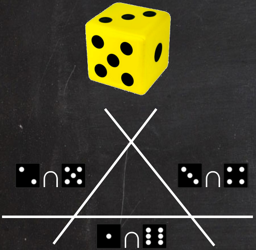
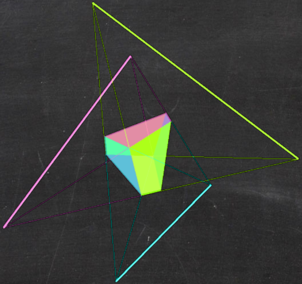


adjoint double plane



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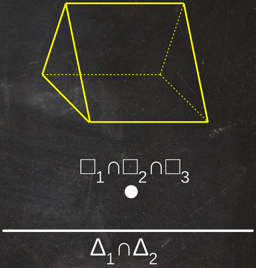

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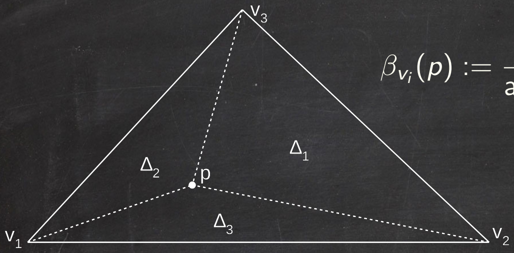
Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

Application 1: Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

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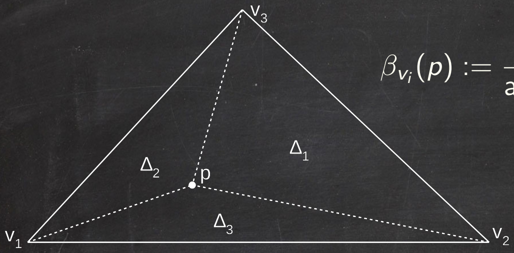

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Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

(i) $\forall u \in V(P) : \beta_u(p) > 0$,

(ii) $\sum_{u \in V(P)} \beta_u(p) = 1$, and

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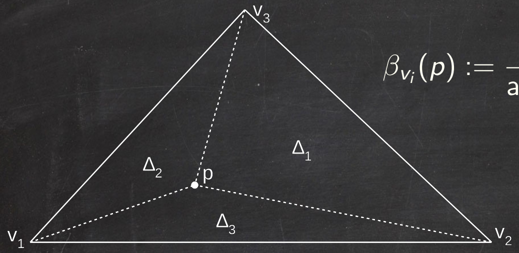

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Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!

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The **Wachspress coordinates**

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For other GBCs and applications of GBCs (e.g., mesh parameterizations in geometric modelling, deformations in computer graphics, or polyhedral FEM): [Floater: Generalized barycentric coordinates and applications, Acta Numerica 24 (2015)]

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

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Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

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Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}}\, m_{\mathcal{I}}(P)\, t^{\mathcal{I}} = \frac{\mathrm{adj_P(t)}}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

where $c_{\mathcal{I}} := \binom{i_1+i_2+\dots+i_n+n}{i_1, i_2, \dots, i_n, n}$.

Application 3: Segre Classes of Monomial Schemes

Aluffi

- V: smooth variety
- $X_1, \ldots, X_n$: smooth hypersurfacs meeting with normal crossings in V

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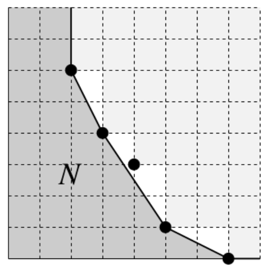
$\mathcal{A} = \{(2,6), (3,4), (4,3), (5,1), (7,0)\}$



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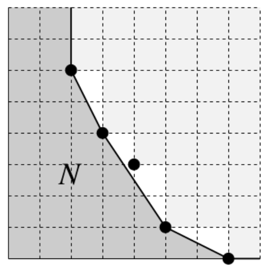

Theorem (Aluffi, (Harris, K., Ranestad))

The Segre class of $S_{\mathcal{A}}$ in the Chow ring of V is

$$\frac{n!\, X_1 \cdots X_n \, \mathrm{adj}_{N_{\mathcal{A}}}(-X)}{\prod_{v \in V(N_{\mathcal{A}})} \ell_v(-X)}$$

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_\mathcal{P}$: simple hyperplane arrangement spanned by facets of P

Idea:

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P \arrow[r, rightsquigarrow] & \mathcal{H}_P \arrow[r, rightsquigarrow] & D \arrow[u, hook] \arrow[r, rightsquigarrow] & \tilde{D} \arrow[u, hook] & \text{smooth}
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When can we find a polytopal hypersurface D?

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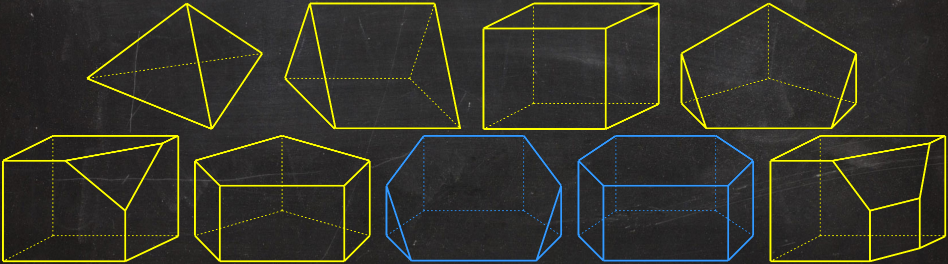
Theorem (K., Ranestad)

Let $\mathcal{C}$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let P be a general polytope of type $\mathcal{C}$. There is a polytopal surface D iff $\mathcal{C}$ is one of:



In that case, the general D is either an elliptic surface or a K3-surface.

Thanks for your attention