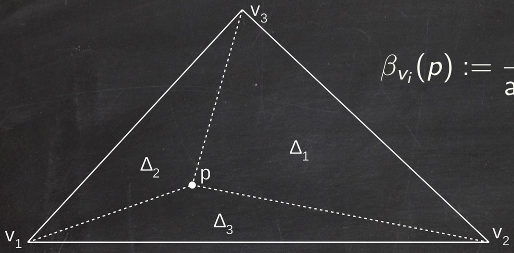
Projective geometry of Wachspress coordinates

Kathlén Kohn
ICERM (Brown University) & Universitetet i Oslo

joint work with Kristian Ranestad (Universitetet i Oslo)

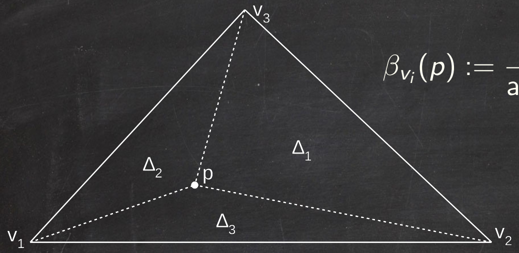
Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

Definition

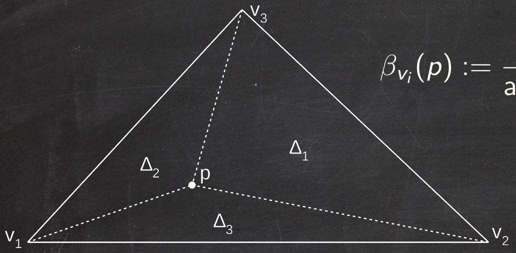
Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

(i) $\forall u \in V(P) : \beta_u(p) > 0$,

(ii) $\sum_{u \in V(P)} \beta_u(p) = 1$, and

(iii) $\sum_{u \in V(P)} \beta_u(p) u = p$.

Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

Definition

Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

(i) $\forall u \in V(P) : \beta_u(p) > 0$,

(ii) $\sum_{u \in V(P)} \beta_u(p) = 1$, and

(iii) $\sum_{u \in V(P)} \beta_u(p) u = p$.

Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!

Barycentric Coordinates

Examples of generalized barycentric coordinates for arbitrary polytopes:

- mean value coordinates
- Wachspress coordinates

Barycentric Coordinates

Examples of generalized barycentric coordinates for arbitrary polytopes:

- mean value coordinates
- Wachspress coordinates

Applications of generalized barycentric coordinates include:

- mesh parameterizations in geometric modelling
- deformations in computer graphics
- polyhedral finite element methods

Barycentric Coordinates

Examples of generalized barycentric coordinates for arbitrary polytopes:

- mean value coordinates
- **Wachspress coordinates**

Applications of generalized barycentric coordinates include:

- mesh parameterizations in geometric modelling
- deformations in computer graphics
- polyhedral finite element methods

The Wachspress coordinates are the unique generalized barycentric coordinates which are rational functions of minimal degree.

The Adjoint of a Polygon

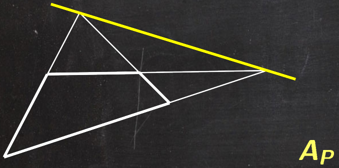
Wachspress (1975)

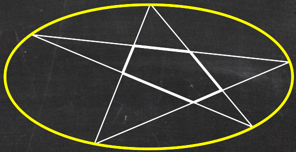
The Adjoint of a Polygon

Wachspress (1975)

Definition

The **adjoint** $\boldsymbol{A_P}$ of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.





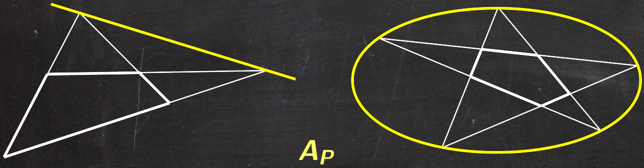
$(\deg A_P = |V(P)| - 3)$

The Adjoint of a Polygon

Wachspress (1975)

Definition

The **adjoint** $\boldsymbol{A_P}$ of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.



$(\deg A_P = |V(P)| - 3)$

Generalization to higher-dimensional polytopes?

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition
$$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

Theorem (Warren)

$\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$*. So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

Theorem (Warren)

I $\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$. *So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

II *If* P *is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.
(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ *dual polytope of* P*)*

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
- $\tau(P)$: triangulation of P using only the vertices of P

Definition $$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma\in\tau(P)} \mathrm{vol}(\sigma) \prod_{v\in V(P)\setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

Theorem (Warren)

I $\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$*. So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

II *If P is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.

(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ *dual polytope of P)*

Geometric definition using a vanishing condition à la Wachspress?

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P

The Adjoint of a Polytope

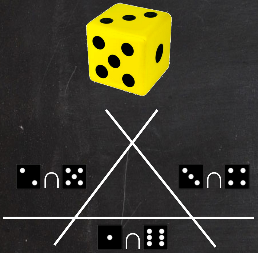
- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

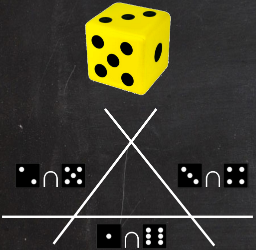
The Adjoint of a Polytope

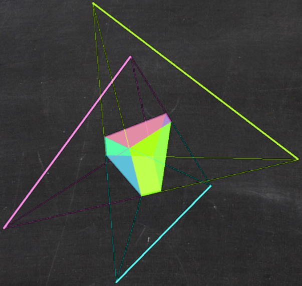
- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P



The Adjoint of a Polytope

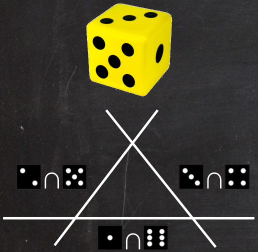
- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

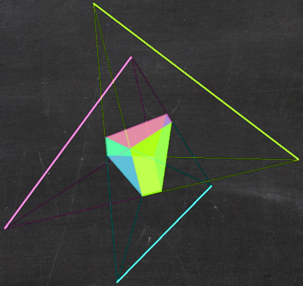


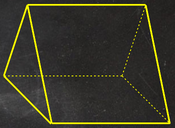


The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

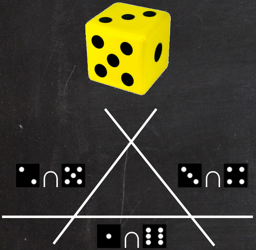


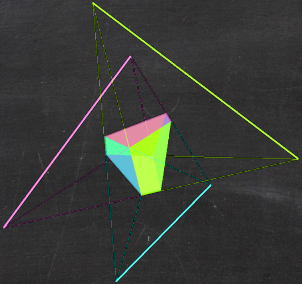


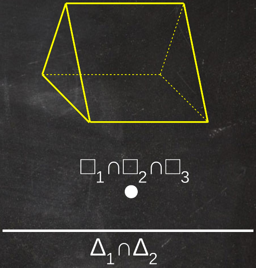


The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

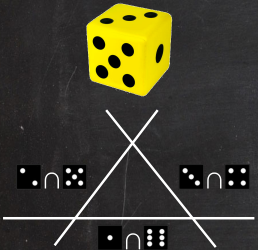


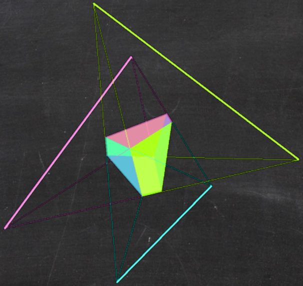


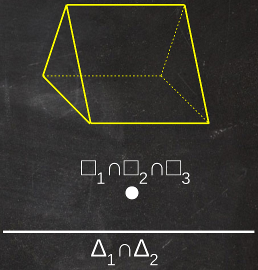


The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





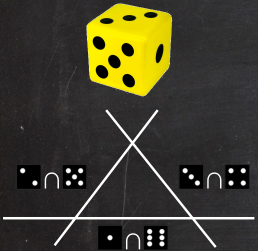


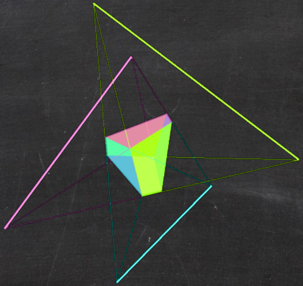
Theorem (K., Ranestad)

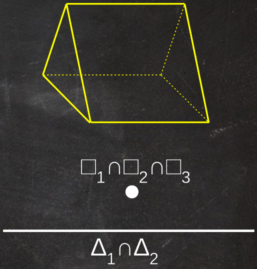
If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





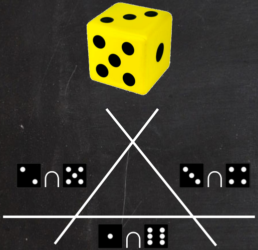
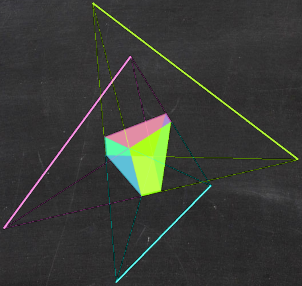
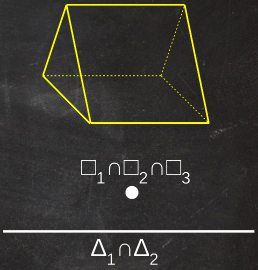


Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

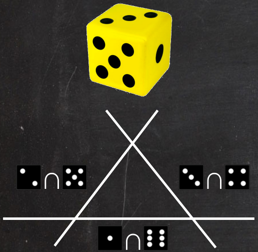




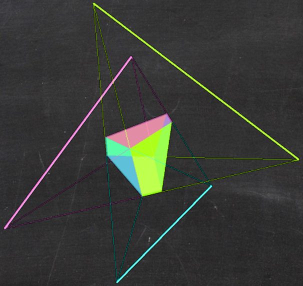
Theorem (K., Ranestad)

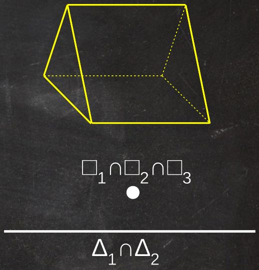
If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d-n-1$ passing through $\mathcal{R}_P$. A_P is called the ***adjoint*** *of P.*

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P







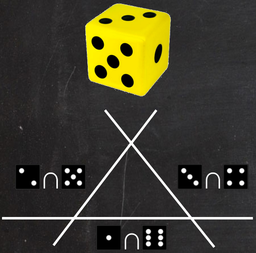
adjoint plane

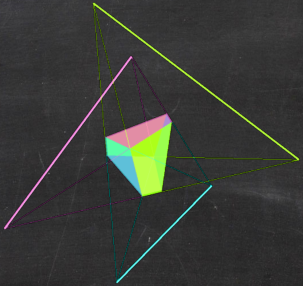
Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d-n-1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

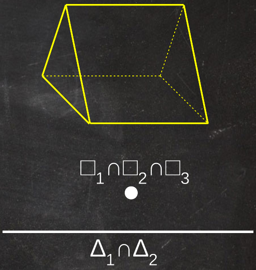
The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P





adjoint quadric surface



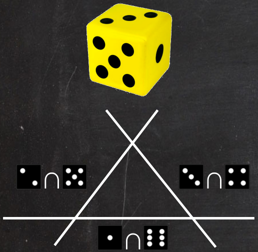
adjoint plane

Theorem (K., Ranestad)

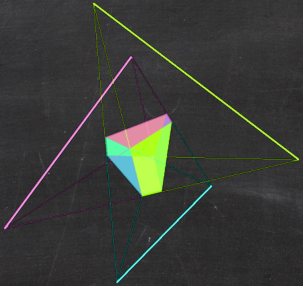
If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

The Adjoint of a Polytope

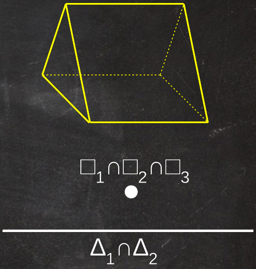
- P: polytope in $\mathbb{P}^n$ **with d facets**
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P



adjoint double plane



adjoint quadric surface



adjoint plane

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d-n-1$ passing through $\mathcal{R}_P$.
A_P is called the **adjoint** *of P.*

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d-n-1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

The Adjoint of a Polytope

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P$: **residual arrangement** of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any of face of P

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes), there is a unique hypersurface A_P in $\mathbb{P}^n$ of degree $d-n-1$ passing through $\mathcal{R}_P$. A_P is called the **adjoint** *of P.*

Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

Wachspress Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

Wachspress Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

$$V(P) \overset{1:1}{\longleftrightarrow} \mathcal{F}(P^*)$$
$$v \longmapsto F_v$$

Wachspress Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

$$V(P) \overset{1:1}{\longleftrightarrow} \mathcal{F}(P^*) \qquad\qquad \mathcal{F}(P) \overset{1:1}{\longleftrightarrow} V(P^*)$$

$$v \longmapsto F_v \qquad\qquad F \longmapsto v_F$$

Wachspress Coordinates

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of P

$$V(P) \overset{1:1}{\longleftrightarrow} \mathcal{F}(P^*) \qquad\qquad \mathcal{F}(P) \overset{1:1}{\longleftrightarrow} V(P^*)$$

$$v \longmapsto F_v \qquad\qquad F \longmapsto v_F$$

Definition (Warren)

The **Wachspress coordinates** of P are

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}.$$

Wachspress Map

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}.$$

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P

Wachspress Map

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}.$$

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P

The numerators of the Wachspress coordinates define the **Wachspress map**:

$$\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)|-1},$$

$$t \longmapsto$$

Wachspress Map

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\operatorname{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\operatorname{adj}_{P^*}(t)}.$$

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P

The numerators of the Wachspress coordinates define the **Wachspress map**:

$$\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)|-1},$$
$$t \longmapsto \left(\prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_F(t) \right)_{u \in V(P)}$$

where ℓ_F is a homogeneous linear equation defining the hyperplane $\operatorname{span}\{F\}$.

Wachspress Map

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- Wachspress map: $\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)|-1}$,

$$t \longmapsto \left(\prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_F(t) \right)_{u \in V(P)}$$

Theorem (K., Ranestad)

The base locus of the Wachspress map ω_P is the residual arrangement $\mathcal{R}_P$.

Wachspress Map

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- Wachspress map: $\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)|-1}$,

$$t \longmapsto \left(\prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_F(t) \right)_{u \in V(P)}$$

Theorem (K., Ranestad)

The base locus of the Wachspress map ω_P is the residual arrangement $\mathcal{R}_P$.

$$\Rightarrow \forall u \in V(P) : \omega_{P,u} \in \boldsymbol{\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d-n))}$$

Wachspress Map

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- Wachspress map: $\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)|-1}$,

$$t \longmapsto \left(\prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_F(t) \right)_{u \in V(P)}$$

Theorem (K., Ranestad)

The base locus of the Wachspress map ω_P is the residual arrangement $\mathcal{R}_P$.

$$\Rightarrow \forall u \in V(P) : \omega_{P,u} \in \boldsymbol{\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d-n))}$$

Theorem (K., Ranestad)

$\dim \Omega_P = |V(P)|$, *so* $\{\omega_{P,u} \mid u \in V(P)\}$ *is a basis of* Ω_P.

Wachspress Map

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- Wachspress map: $\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)|-1}$,

$$t \longmapsto \left(\prod_{F \in \mathcal{F}(P) \colon u \notin F} \ell_F(t) \right)_{u \in V(P)}$$

Theorem (K., Ranestad)

The base locus of the Wachspress map ω_P is the residual arrangement $\mathcal{R}_P$.

$$\Rightarrow \forall u \in V(P) : \omega_{P,u} \in \boldsymbol{\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d-n))}$$

Theorem (K., Ranestad)

$\dim \Omega_P = |V(P)|$, *so* $\{\omega_{P,u} \mid u \in V(P)\}$ *is a basis of* Ω_P.

$$\Rightarrow \boldsymbol{\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}(\Omega_P^*) \cong \mathbb{P}^{|V(P)|-1}}$$

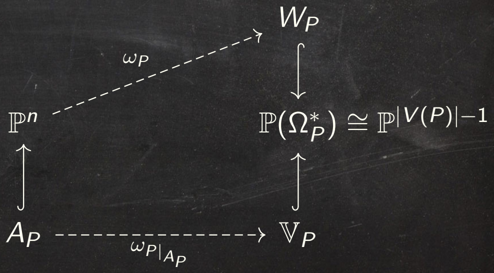
Wachspress Map

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- $\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d-n))$
- $W_P := \overline{\omega_P(\mathbb{P}^n)}$ is the **Wachspress variety**

$$\begin{array}{ccc} & & W_P \\ & \overset{\omega_P}{\nearrow} & \downarrow \\ \mathbb{P}^n & & \mathbb{P}(\Omega_P^*) \cong \mathbb{P}^{|V(P)|-1} \end{array}$$

Wachspress Map

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- $\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d-n))$
- $W_P := \overline{\omega_P(\mathbb{P}^n)}$ is the **Wachspress variety**
- $\mathbb{V}_P := \mathrm{span}\{\omega_P(A_P)\}$



Wachspress Map

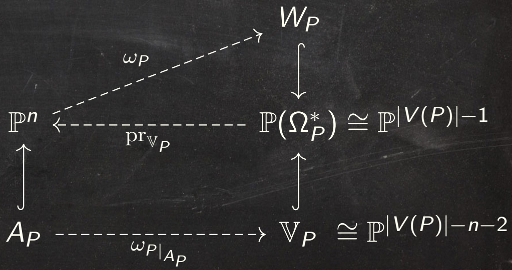
- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- $\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d-n))$
- $W_P := \overline{\omega_P(\mathbb{P}^n)}$ is the **Wachspress variety**
- $\mathbb{V}_P := \mathrm{span}\{\omega_P(A_P)\}$

Theorem (K., Ranestad)

$\dim \mathbb{V}_P = |V(P)| - n - 2.$

The projection

$\mathrm{pr}_{\mathbb{V}_P} : \mathbb{P}(\Omega_P^*) \dashrightarrow \mathbb{P}^n$ *from* $\mathbb{V}_P$

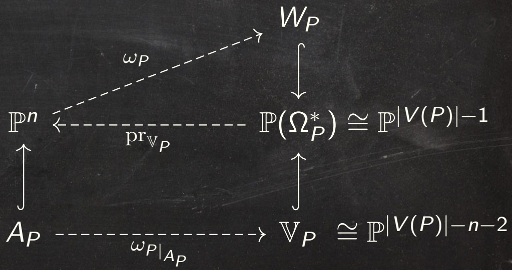


Wachspress Map

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- $\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d-n))$
- $W_P := \overline{\omega_P(\mathbb{P}^n)}$ is the **Wachspress variety**
- $\mathbb{V}_P := \mathrm{span}\{\omega_P(A_P)\}$

Theorem (K., Ranestad)
$\dim \mathbb{V}_P = |V(P)| - n - 2.$

The projection $\mathrm{pr}_{\mathbb{V}_P} : \mathbb{P}(\Omega_P^*) \dashrightarrow \mathbb{P}^n$ *from* $\mathbb{V}_P$ *restricted to the Wachspress variety* W_P *is the inverse of the Wachspress map* ω_P.



Wachspress Surfaces

$$\begin{array}{ccccc} & & & & W_P \\ & & \overset{\omega_P}{\nearrow} & & \downarrow \\ \mathbb{P}^2 & & \xleftarrow{\quad \mathrm{pr}_{\mathbb{V}_P} \quad} & & \mathbb{P}(\Omega_P^*) \cong \mathbb{P}^{d-1} \\ \uparrow & & & & \uparrow \\ A_P & & \xrightarrow{\quad \omega_{P|_{A_P}} \quad} & & \mathbb{V}_P \cong \mathbb{P}^{d-4} \end{array}$$

Theorem (Irving, Schenck)

Let P be a d-gon in $\mathbb{P}^2$.

Wachspress Surfaces

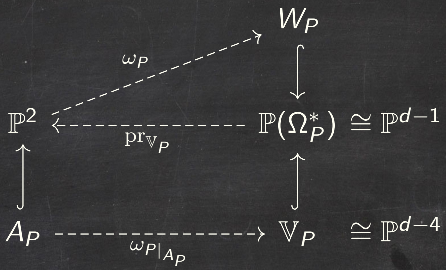
$$
\begin{array}{ccccc}
 & & W_P & & \\
 & \overset{\omega_P}{\nearrow} & \downarrow & & \\
\mathbb{P}^2 & \xleftarrow{\ \mathrm{pr}_{\mathbb{V}_P}\ } & \mathbb{P}(\Omega_P^*) & \cong & \mathbb{P}^{d-1} \\
\uparrow & & \uparrow & & \\
A_P & \xrightarrow{\ \omega_{P|_{A_P}}\ } & \mathbb{V}_P & \cong & \mathbb{P}^{d-4}
\end{array}
$$

Theorem (Irving, Schenck)

Let P be a d-gon in $\mathbb{P}^2$.

- *The Wachspress variety W_P is a surface of degree $\binom{d-2}{2}+1$.*

Wachspress Surfaces

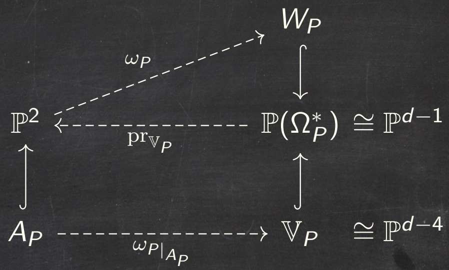


Theorem (Irving, Schenck)

Let P be a d-gon in $\mathbb{P}^2$.

- *The Wachspress variety W_P is a surface of degree $\binom{d-2}{2} + 1$.*
- *The image of the adjoint curve A_P under ω_P is a curve of degree $\binom{d-3}{2}$, if $d > 4$.*

Wachspress Surfaces



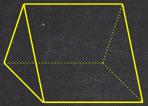
Theorem (Irving, Schenck)

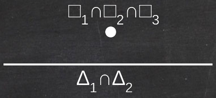
Let P be a d-gon in $\mathbb{P}^2$.

- *The Wachspress variety W_P is a surface of degree $\binom{d-2}{2} + 1$.*
- *The image of the adjoint curve A_P under ω_P is a curve of degree $\binom{d-3}{2}$, if $d > 4$.*
- *If $d = 4$, the image of the adjoint line A_P is a point.*

Wachspress Threefolds

P



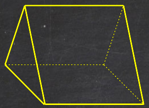


$\mathcal{R}_P$

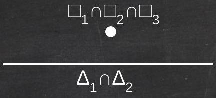
A_P adjoint plane

Wachspress Threefolds

P



$\mathcal{R}_P$

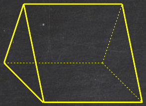


A_P adjoint plane

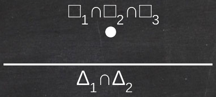
ω_P $(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$

Wachspress Threefolds

P



$\mathcal{R}_P$



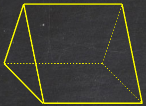
A_P adjoint plane

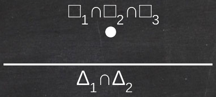
ω_P $(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$

W_P $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

Wachspress Threefolds

P





$\mathcal{R}_P$

A_P adjoint plane

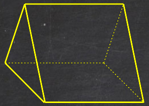
ω_P $(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$

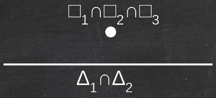
W_P $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

$\omega_{P|_{A_P}}$ projection from point

Wachspress Threefolds

P



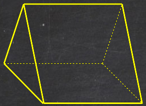


$\mathcal{R}_P$

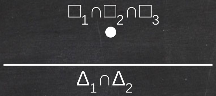
A_P	adjoint plane
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$
$\omega_{P\|_{A_P}}$	projection from point
$\omega_P(A_P)$	line

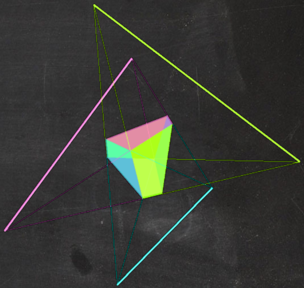
Wachspress Threefolds

P



$\mathcal{R}_P$

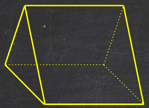


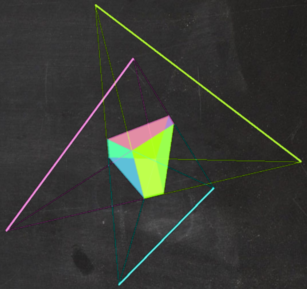


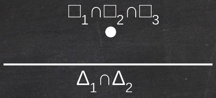
A_P	adjoint plane	adjoint quadric surface
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$	
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	
$\omega_{P\vert_{A_P}}$	projection from point	
$\omega_P(A_P)$	line	

Wachspress Threefolds

P



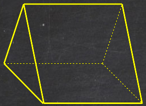
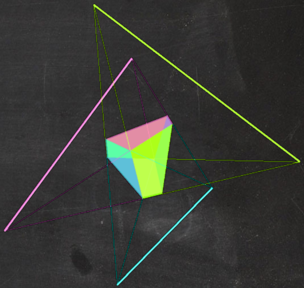
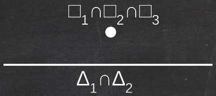




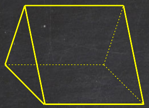
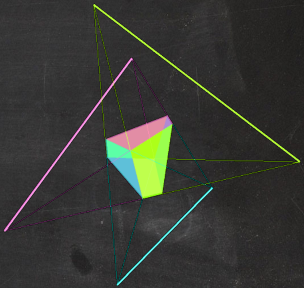
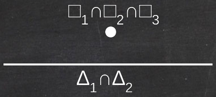
$\mathcal{R}_P$

A_P	adjoint plane	adjoint quadric surface
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$	$(\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4)$
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	
$\omega_{P\|_{A_P}}$	projection from point	
$\omega_P(A_P)$	line	

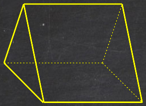
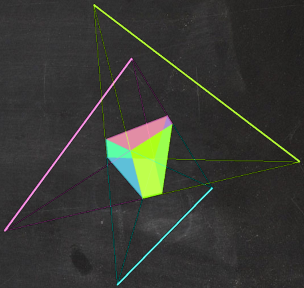
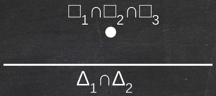
Wachspress Threefolds

P		
$\mathcal{R}_P$		
A_P	adjoint plane	adjoint quadric surface
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$	$(\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4)$
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$
$\omega_{P\|_{A_P}}$	projection from point	
$\omega_P(A_P)$	line	

Wachspress Threefolds

P		
$\mathcal{R}_P$		
A_P	adjoint plane	adjoint quadric surface
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$	$(\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4)$
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$
$\omega_{P\|_{A_P}}$	projection from point	contracts ruling of lines
$\omega_P(A_P)$	line	

Wachspress Threefolds

P		
$\mathcal{R}_P$		
A_P	adjoint plane	adjoint quadric surface
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$	$(\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4)$
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$
$\omega_{P\|_{A_P}}$	projection from point	contracts ruling of lines
$\omega_P(A_P)$	line	twisted cubic curve

Wachspress Threefolds

- P: polytope in $\mathbb{P}^3$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- a: number of isolated points in $\mathcal{R}_P$
- b: number of double points in $\mathcal{R}_P$
- c: number of triple points in $\mathcal{R}_P$

Wachspress Threefolds

- P: polytope in $\mathbb{P}^3$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- a: number of isolated points in $\mathcal{R}_P$
- b: number of double points in $\mathcal{R}_P$
- c: number of triple points in $\mathcal{R}_P$

Proposition (K., Ranestad)

The Wachspress variety $W_P \subset \mathbb{P}^{2d-5}$ is a threefold of degree

$$2b + 4c - a - \frac{1}{2}(d-3)(d^2 - 11d + 26) = b + 2c + 1 - \frac{1}{6}(d-3)(d-4)(d-11)$$

and sectional genus $b + 2c + 1 + \frac{1}{2}(d-3)(d-6)$.

Wachspress Threefolds

- P: polytope in $\mathbb{P}^3$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- a: number of isolated points in $\mathcal{R}_P$
- b: number of double points in $\mathcal{R}_P$
- c: number of triple points in $\mathcal{R}_P$

Proposition (K., Ranestad)

The Wachspress variety $W_P \subset \mathbb{P}^{2d-5}$ is a threefold of degree

$$2b + 4c - a - \frac{1}{2}(d-3)(d^2 - 11d + 26) = b + 2c + 1 - \frac{1}{6}(d-3)(d-4)(d-11)$$

and sectional genus $b + 2c + 1 + \frac{1}{2}(d-3)(d-6)$.
The image of the adjoint surface A_P under ω_P is a surface iff P is neither a tetrahedron, a triangular prism nor a cube.

Wachspress Threefolds

- P: polytope in $\mathbb{P}^3$ with d facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of P
- a: number of isolated points in $\mathcal{R}_P$
- b: number of double points in $\mathcal{R}_P$
- c: number of triple points in $\mathcal{R}_P$

Proposition (K., Ranestad)

The Wachspress variety $W_P \subset \mathbb{P}^{2d-5}$ is a threefold of degree

$$2b + 4c - a - \frac{1}{2}(d-3)(d^2 - 11d + 26) = b + 2c + 1 - \frac{1}{6}(d-3)(d-4)(d-11)$$

and sectional genus $b + 2c + 1 + \frac{1}{2}(d-3)(d-6)$.
The image of the adjoint surface A_P under ω_P is a surface iff P is neither a tetrahedron, a triangular prism nor a cube. In that case, its degree is

$$2b + 4c - a - \frac{1}{2}(d-3)(d-4)(d-6) = b + 2c + 1 - \frac{1}{6}(d-3)(d^2 - 12d + 38)$$

and its sectional genus is $b + 2c + 1 - \frac{1}{2}(d-3)(d-4)$.

Why “Adjoint”?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P

Idea:

$$P \rightsquigarrow \mathcal{H}_P$$

hypersurface
of degree d

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$P \rightsquigarrow \mathcal{H}_P \rightsquigarrow D$$

hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_\mathcal{P}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$
\begin{array}{ccccccc}
 & & & & \mathbb{P}^n & \xleftarrow{\text{blowup } \pi} & X \ \text{smooth} \\
 & & & & \uparrow & & \uparrow \\
P & \rightsquigarrow & \mathcal{H}_P & \rightsquigarrow & D & \rightsquigarrow & \tilde{D} \ \text{smooth}
\end{array}
$$

hypersurface of degree d (under $\mathcal{H}_P$)

polytopal hypersurface: hypersurface of degree d, multiplicity c along $\mathcal{R}_P^c$, smooth outside of $\mathcal{R}_P$ (under D)

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$\mathbb{P}^n \xleftarrow{\text{blowup } \pi} X \quad \text{smooth}$$

$$D \hookrightarrow \mathbb{P}^n, \qquad \tilde{D} \hookrightarrow X$$

$$P \rightsquigarrow \mathcal{H}_P \rightsquigarrow D \rightsquigarrow \tilde{D} \quad \text{smooth}$$

$\mathcal{H}_P$: hypersurface of degree d

D: **polytopal hypersurface:** hypersurface of degree d, multiplicity c along $\mathcal{R}_P^c$, smooth outside of $\mathcal{R}_P$

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$$\begin{array}{ccccccc} & & & & \mathbb{P}^n & \xleftarrow{\text{blowup } \pi} & X \quad \text{smooth} \\ & & & & \uparrow & & \uparrow \\ P & \rightsquigarrow & \mathcal{H}_P & \rightsquigarrow & D & \rightsquigarrow & \tilde{D} \quad \text{smooth} \end{array}$$

hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

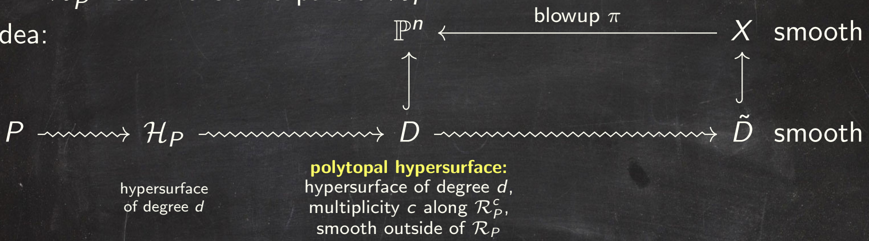
Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Def.: An **adjoint to $\tilde{D}$ in X** is a hypersurface A in X s.t. $[A] = K_X + [\tilde{D}]$.

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:



Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Def.: An **adjoint to $\tilde{D}$ in X** is a hypersurface A in X s.t. $[A] = K_X + [\tilde{D}]$.

Proposition (K., Ranestad)

$\tilde{D}$ has a unique adjoint A in X, and thus a unique canonical divisor: $A \cap \tilde{D}$. Moreover, $\pi(A) = A_P$.

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P
- $\mathcal{R}_P^c$: codimension-c part of $\mathcal{R}_P$

Idea:

$\mathbb{P}^n \xleftarrow{\text{blowup } \pi} X$ smooth

$P \rightsquigarrow \mathcal{H}_P \rightsquigarrow D \rightsquigarrow \tilde{D}$ smooth

($D \hookrightarrow \mathbb{P}^n$, $\tilde{D} \hookrightarrow X$)

$\mathcal{H}_P$: hypersurface of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
smooth outside of $\mathcal{R}_P$

When can we find
a polytopal hypersurface D?

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

Def.: An **adjoint to $\tilde{D}$ in X** is a hypersurface A in X s.t. $[A] = K_X + [\tilde{D}]$.

Proposition (K., Ranestad)

$\tilde{D}$ has a unique adjoint A in X, and thus a unique canonical divisor: $A \cap \tilde{D}$.
Moreover, $\pi(A) = A_P$.

Polytopal Hypersurfaces

Proposition (K., Ranestad)

Let P be a general d-gon in $\mathbb{P}^2$. There is a polygonal curve D iff $d \leq 6$. In that case, D is an elliptic curve.

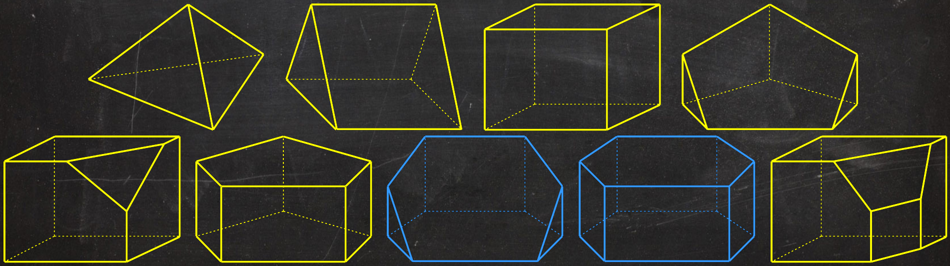
Polytopal Hypersurfaces

Proposition (K., Ranestad)

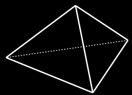
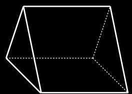
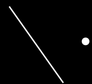
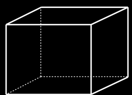
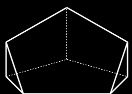
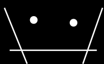
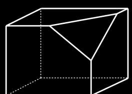
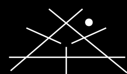
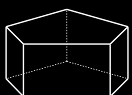
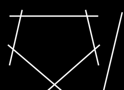
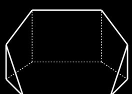
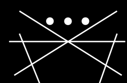
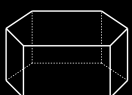
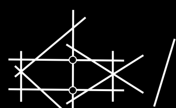
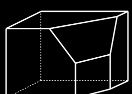
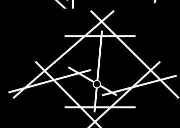
Let P be a general d-gon in $\mathbb{P}^2$. There is a polygonal curve D iff $d \leq 6$. In that case, D is an elliptic curve.

Theorem (K., Ranestad)

Let $\mathcal{C}$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let P be a general polytope of type $\mathcal{C}$. There is a polytopal surface D iff $\mathcal{C}$ is one of:



In that case, the general D is either an elliptic surface or a K3-surface.

comb. type	facet sizes	$\mathcal{R}_P$	(a,b,c)	W_P (deg., sec. genus)	$\overline{w_P(A_P)}$ (deg., sec. genus)	$\dim \Gamma_P$	$\overline{w_P(D)}$ (deg., sec. genus)
	3 3 3 3		$(0,0,0)$	$\mathbb{P}^3$ $(1,0)$	0	34	minimal K3 (smooth quartic in $\mathbb{P}^3$)
	4 4 4 3 3		$(1,0,0)$	$\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ $(3,0)$	line	23	minimal K3 $(8,5)$
	4 4 4 4 4 4		$(0,0,0)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ $(6,1)$	twisted cubic curve	26	minimal K3 $(12,7)$
	5 5 4 4 3 3		$(2,2,0)$	$W_P \subset \mathbb{P}^7$ $(8,3)$	quadric surface $(2,0)$	17	non-minimal K3 $(14,9)$
	5 5 5 4 4 4 3		$(1,6,0)$	$W_P \subset \mathbb{P}^9$ $(15,9)$	del Pezzo surface in $\mathbb{P}^5$ $(5,1)$	7	non-minimal K3 $(19,12)$
	5 5 4 4 4 4 4		$(0,5,0)$	Fano 3-fold in $\mathbb{P}^9$ $(14,8)$	rational scroll in $\mathbb{P}^5$ $(4,0)$	12	non-minimal K3 $(18,11)$
	6 6 4 4 4 3 3		$(3,6,1)$	$W_P \subset \mathbb{P}^9$ $(17,11)$	rational elliptic surface in $\mathbb{P}^5$ $(7,3)$	4	minimal elliptic $(22,15)$
	6 6 4 4 4 4 4 4		$(0,12,2)$	$W_P \subset \mathbb{P}^{11}$ $(27,22)$	elliptic K3-surface in $\mathbb{P}^7$ $(12,7)$	3	minimal elliptic $(26,17)$
	5 5 5 5 4 4 4 4		$(0,16,0)$	$W_P \subset \mathbb{P}^{11}$ $(27,22)$	K3-surface in $\mathbb{P}^7$ $(12,7)$	1	non-minimal K3 $(24,15)$