The Adjoint Polynomial of a Polytope

Kathlén Kohn

joints works with Kristian Ranestad (Universitetet i Oslo),
Boris Shapiro (Stockholms universitet) & Bernd Sturmfels (MPI MiS Leipzig / UC Berkeley)

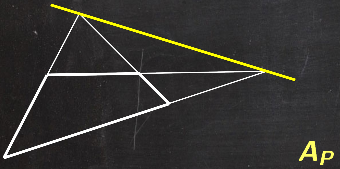
November 6, 2019

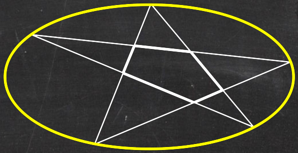
The Adjoint of a Polygon

Wachspress (1975)

Definition

The **adjoint A_P** of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.





$(\deg A_P = |V(P)| - 3)$

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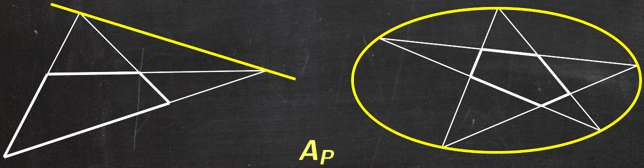

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Generalization to higher-dimensional polytopes?

The Adjoint of a Polytope

Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
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$$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

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Theorem (Warren)

$\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$*. So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

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II *If P is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.

(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ *dual polytope of P)*

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Geometric definition using a vanishing condition à la Wachspress?

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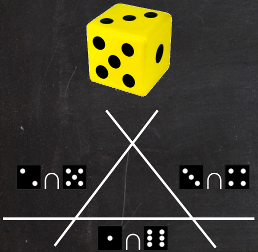

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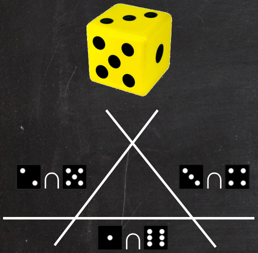

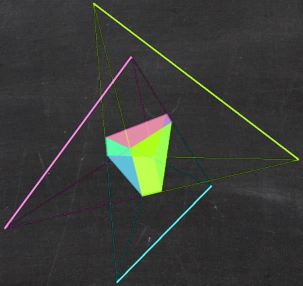

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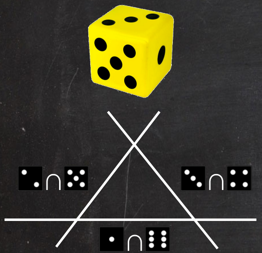

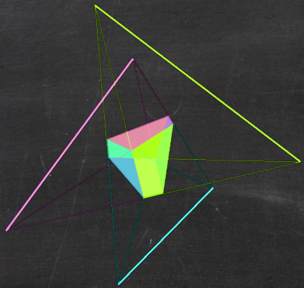

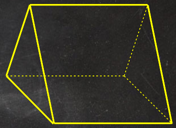

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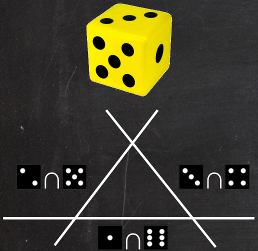

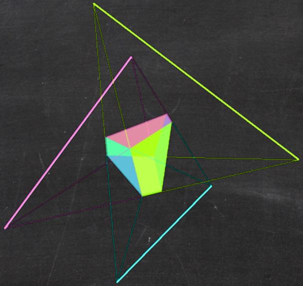

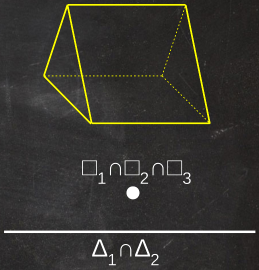

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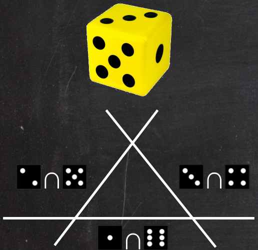

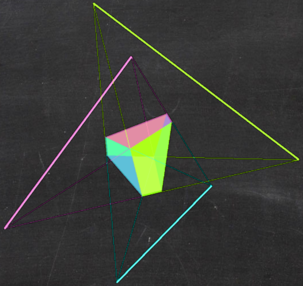

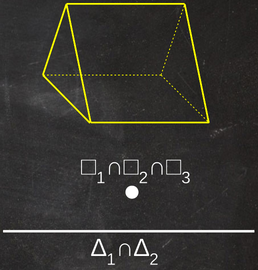

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

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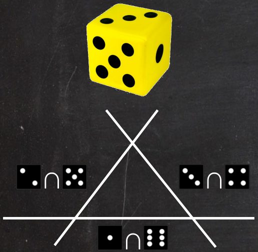

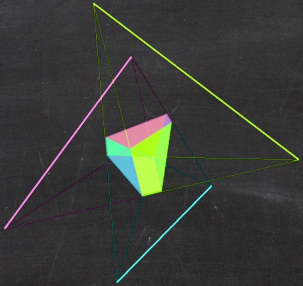

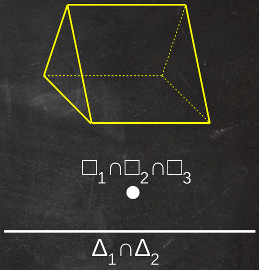

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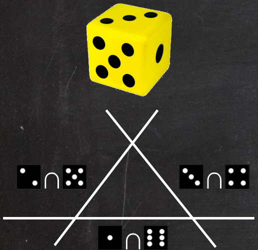

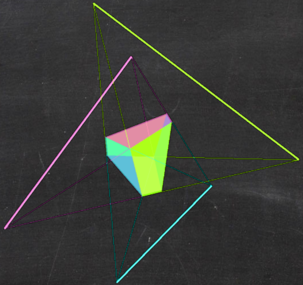

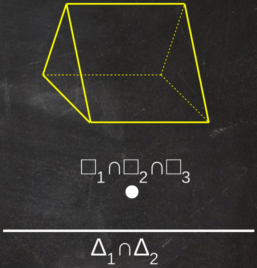

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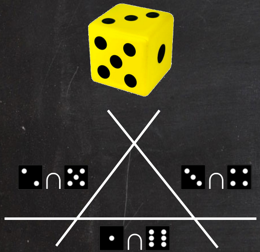

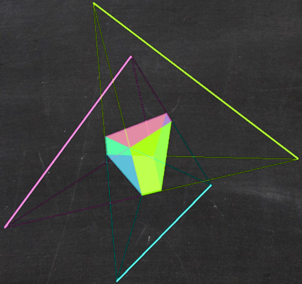

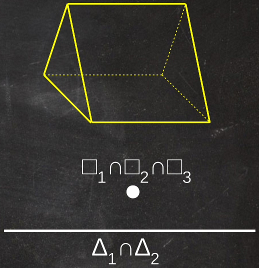

adjoint plane

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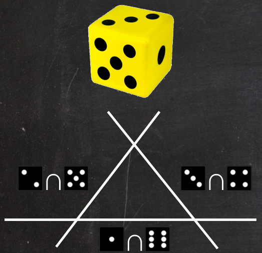
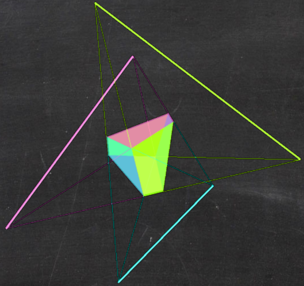
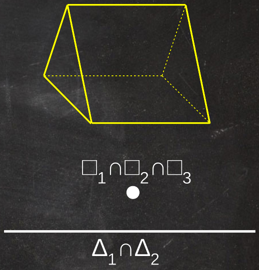




adjoint quadric surface



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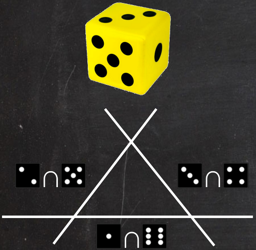
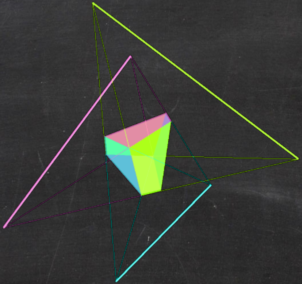


adjoint double plane



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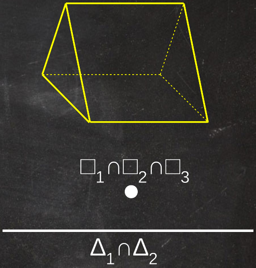

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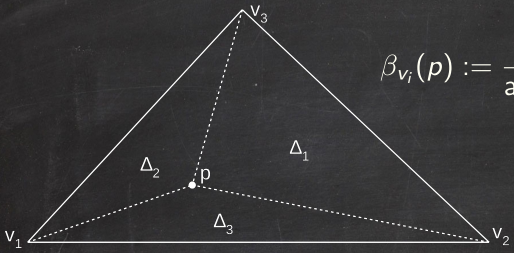
Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

Application 1: Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

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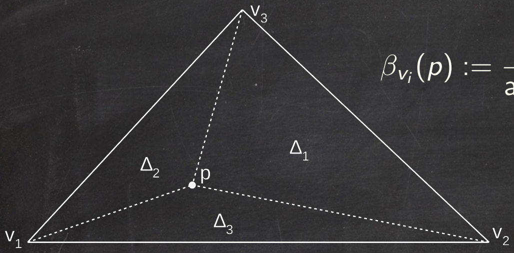

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(i) $\forall u \in V(P) : \beta_u(p) > 0$,

(ii) $\sum_{u \in V(P)} \beta_u(p) = 1$, and

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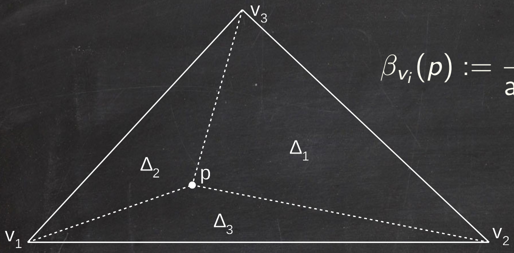

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Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!

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The **Wachspress coordinates**

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For other GBCs and applications of GBCs (e.g., mesh parameterizations in geometric modelling, deformations in computer graphics, or polyhedral FEM):
[Floater: Generalized barycentric coordinates and applications, Acta Numerica 24 (2015)]

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

- P: convex polytope in $\mathbb{R}^n$
- μ_P: uniform probability distribution on P

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K., Shapiro, Sturmfels

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- μ_P: uniform probability distribution on P
- moments

$$m_{\mathcal{I}}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \dots w_n^{i_n} d\mu_P \quad \text{for } \mathcal{I} = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$$

Application 2: Moments of Probability Distributions

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Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}}\, m_{\mathcal{I}}(P)\, t^{\mathcal{I}} = \frac{\mathrm{adj_P(t)}}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

where $c_{\mathcal{I}} := \binom{i_1+i_2+\dots+i_n+n}{i_1, i_2, \dots, i_n, n}$.

Application 3: Segre Classes of Monomial Schemes

Aluffi

- V: smooth variety
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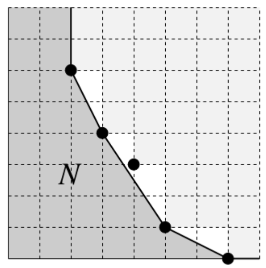
Example: $n = 2$

$\mathcal{A} = \{(2,6),(3,4),(4,3),(5,1),(7,0)\}$



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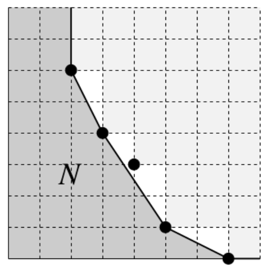

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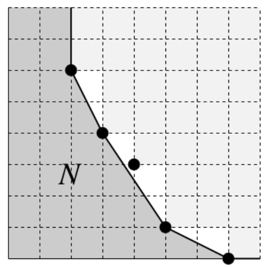

Theorem (Aluffi, (K., Ranestad))

The Segre class of $S_{\mathcal{A}}$ in the Chow ring of V is

$$\frac{n!\, X_1 \cdots X_n \,\mathrm{adj}_{N_{\mathcal{A}}}(-X)}{\prod_{v \in V(N_{\mathcal{A}})} \ell_v(-X)}, \quad \text{if } N_{\mathcal{A}} \text{ is finite.}$$

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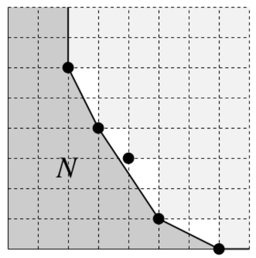

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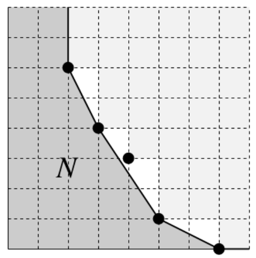

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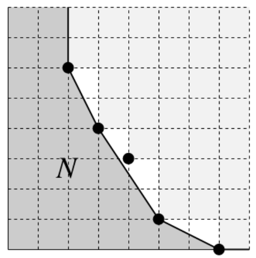

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where

$$\mathrm{adj}_{N_{\mathcal{A}}}(t) = 1 - 15t_1 - 22t_2 + 71t_1^2 + 212t_1t_2 + 95t_2^2 - 105t_1^3 - 476t_1^2t_2 - 511t_1t_2^2 - 84t_2^3.$$

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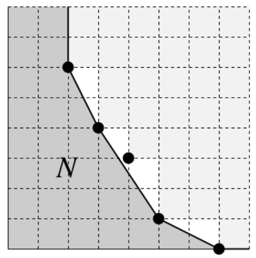

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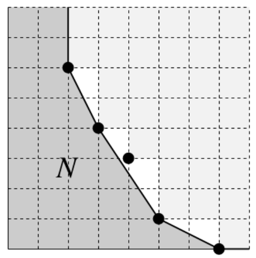

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Open Question: What do they count?

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