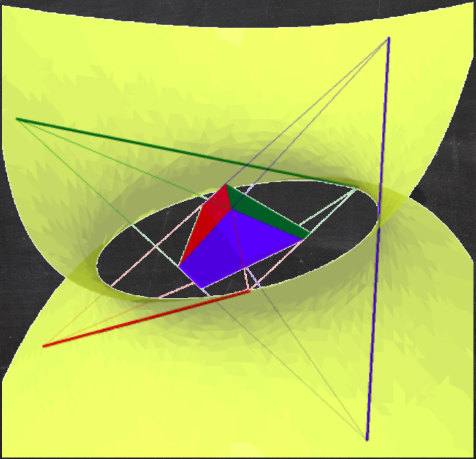
Projective geometry of Wachspress coordinates

Kathlén Kohn
KTH



joint works with Kristian Ranestad (Universitetet i Oslo) /
Boris Shapiro (Stockholms universitet) & Bernd Sturmfels (MPI MiS Leipzig / UC Berkeley)

The Adjoint of a Polygon

Wachspress (1975)

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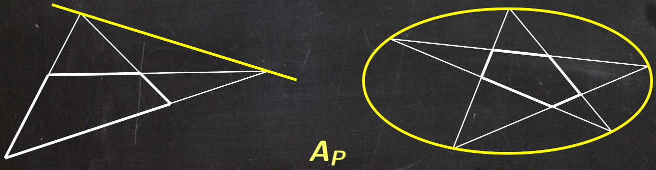
Definition

The **adjoint A_P** of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.



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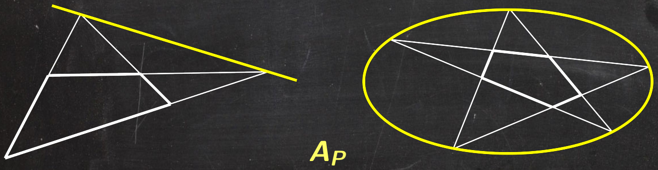

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Generalization to higher-dimensional polytopes?

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- P: convex polytope in $\mathbb{R}^n$
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where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

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Geometric definition using a vanishing condition à la Wachspress?

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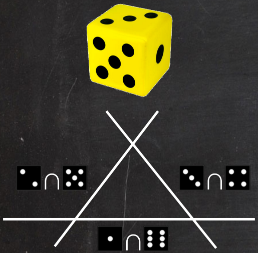

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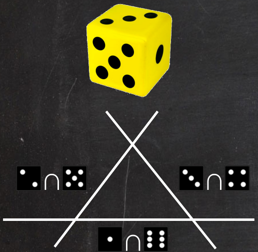

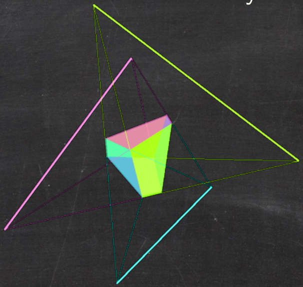

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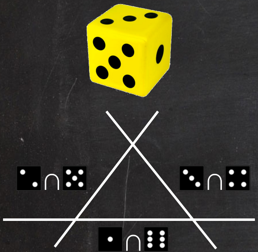

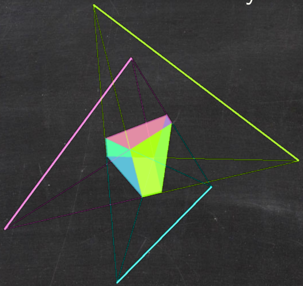

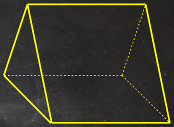

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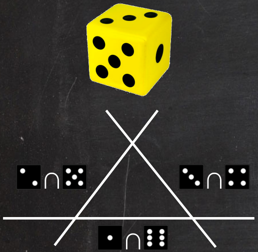

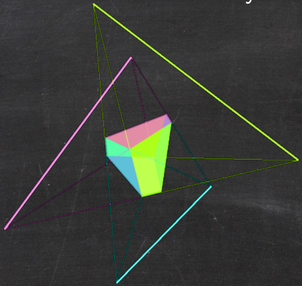

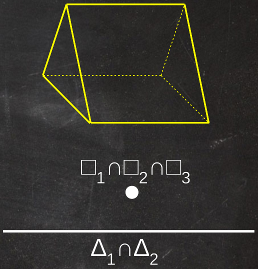

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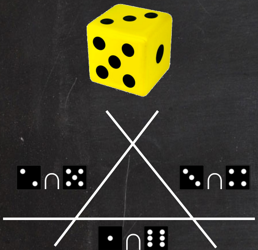

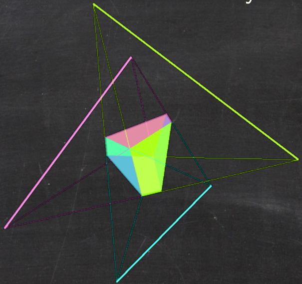

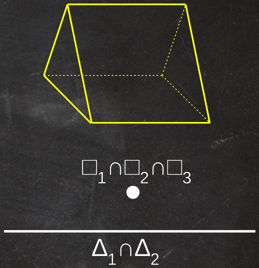

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If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

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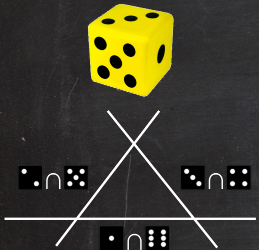

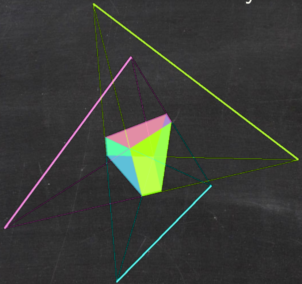

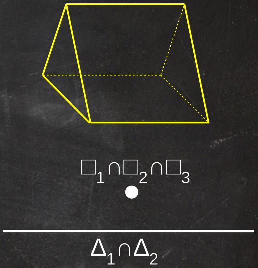

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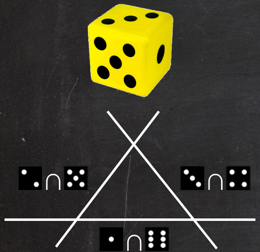

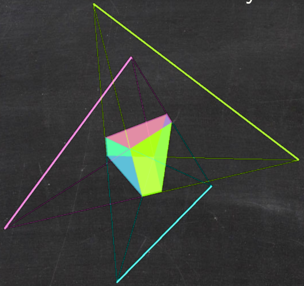

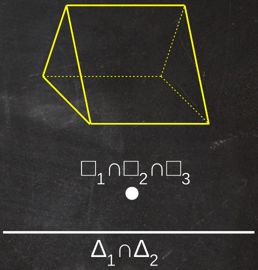

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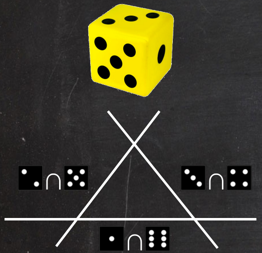

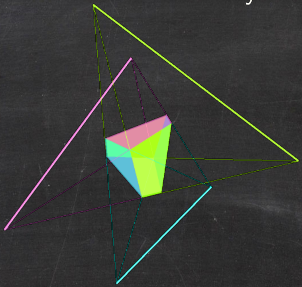

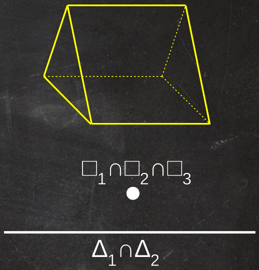

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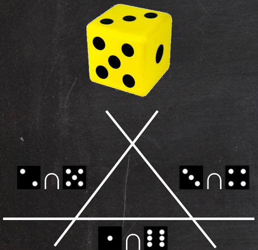
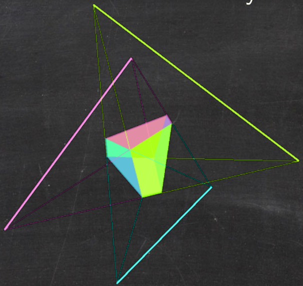
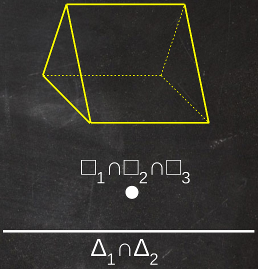




adjoint quadric surface



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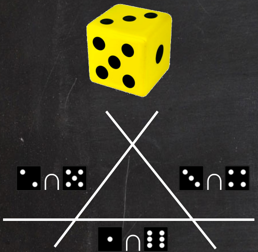

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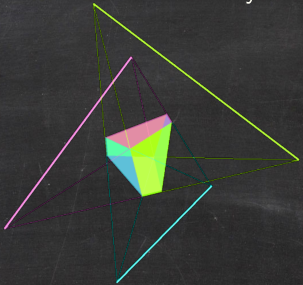

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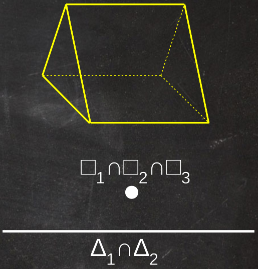

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Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

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- V: smooth variety
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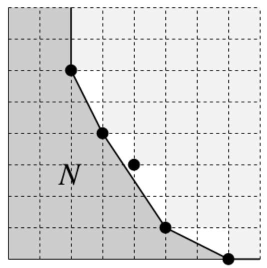
Example: $n = 2$

$\mathcal{A} = \{(2,6), (3,4), (4,3), (5,1), (7,0)\}$



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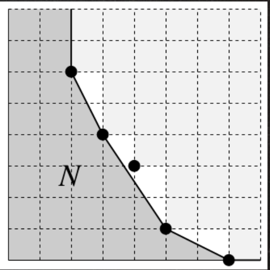

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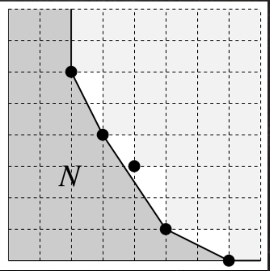

Theorem (Aluffi, (K., Ranestad))

The Segre class of $S_{\mathcal{A}}$ in the Chow ring of V is

$$\frac{n!\, X_1 \cdots X_n \,\mathrm{adj}_{N_{\mathcal{A}}}(-X)}{\prod_{v \in V(N_{\mathcal{A}})} \ell_v(-X)},$$ *if $N_{\mathcal{A}}$ is finite.*

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- $N_{\mathcal{A}}$ may have vertices at ∞ in the direction of the standard basis vectors $e_1, \dots, e_n$

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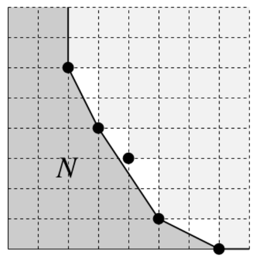
$\mathcal{A} = \{(2,6),(3,4),(4,3),(5,1),(7,0)\}$



Application 1: Segre Classes of Monomial Schemes

Aluffi

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- for vertex v_i at ∞ in direction of e_i: $\ell_{v_i}(t) := -t_i$

Theorem (Aluffi, (K., Ranestad))

The Segre class of $S_{\mathcal{A}}$ in the Chow ring of V is

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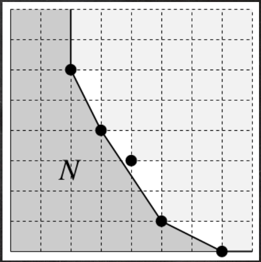

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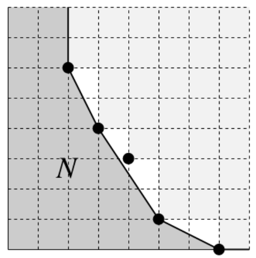

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Example:

$$\frac{2X_1X_2\,\mathrm{adj}_{N_{\mathcal{A}}}(-X_1,-X_2)}{X_2(1+2X_1+6X_2)(1+3X_1+4X_2)(1+5X_1+X_2)(1+7X_1)},$$

where

$$\mathrm{adj}_{N_{\mathcal{A}}}(t) = 1 - 15t_1 - 22t_2 + 71t_1^2 + 212t_1t_2 + 95t_2^2 - 105t_1^3 - 476t_1^2t_2 - 511t_1t_2^2 - 84t_2^3.$$

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

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$$m_{\mathcal{I}}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \ldots w_n^{i_n} d\mu_P \quad \text{for } \mathcal{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$$

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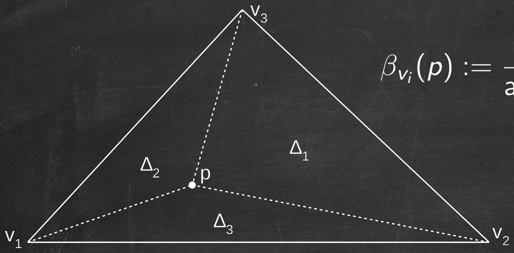
Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}}\, m_{\mathcal{I}}(P)\, t^{\mathcal{I}} = \frac{\mathrm{adj_P(t)}}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

where $c_{\mathcal{I}} := \binom{i_1+i_2+\ldots+i_n+n}{i_1, i_2, \ldots, i_n, n}$.

Application 3: Barycentric Coordinates



$$\beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)}$$

for $i = 1, 2, 3$

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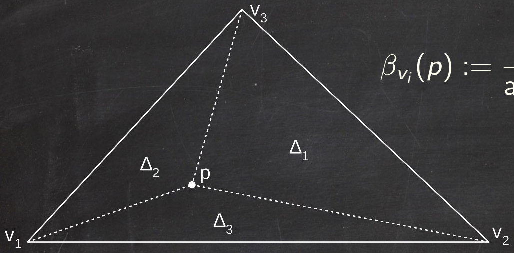

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Definition

Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

- $\forall u \in V(P) : \beta_u(p) > 0$,
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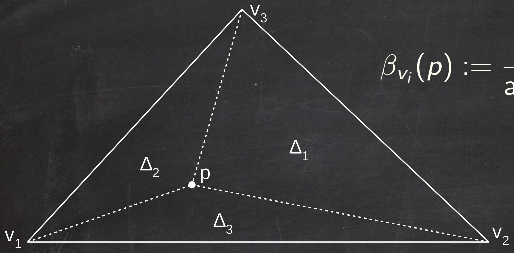

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Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!

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- mean value coordinates
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The Wachspress coordinates are the unique generalized barycentric coordinates which are rational functions of minimal degree.

Wachspress Coordinates

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Definition (Warren)

The **Wachspress coordinates** of P are

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}.$$

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$$\omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)|-1},$$

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where ℓ_F is a homogeneous linear equation defining the hyperplane $\mathrm{span}\{F\}$.

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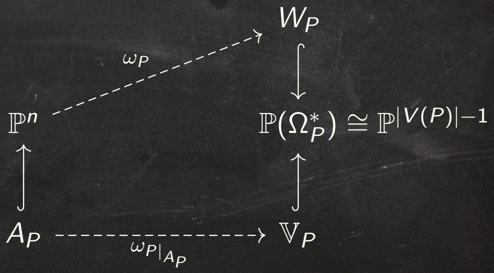

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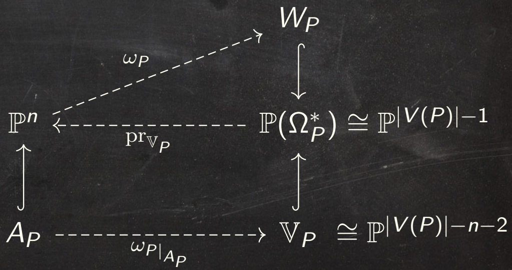
$\dim \mathbb{V}_P = |V(P)| - n - 2.$

The projection
$\mathrm{pr}_{\mathbb{V}_P} : \mathbb{P}(\Omega_P^*) \dashrightarrow \mathbb{P}^n$ *from* $\mathbb{V}_P$



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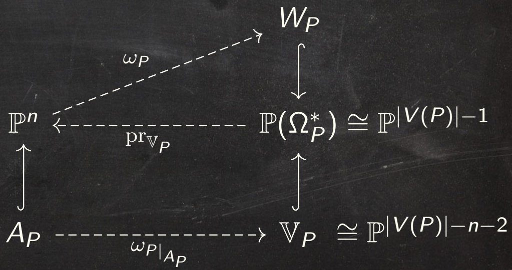
The projection $\mathrm{pr}_{\mathbb{V}_P} : \mathbb{P}(\Omega_P^*) \dashrightarrow \mathbb{P}^n$ from $\mathbb{V}_P$ restricted to the Wachspress variety W_P is the inverse of the Wachspress map ω_P.



Wachspress Surfaces

$$
\begin{array}{ccccc}
 & & & & W_P \\
 & & \overset{\omega_P}{\nearrow} & & \downarrow \\
\mathbb{P}^2 & \xleftarrow{\quad \mathrm{pr}_{\mathbb{V}_P} \quad} & & & \mathbb{P}(\Omega_P^*) \cong \mathbb{P}^{d-1} \\
\uparrow & & & & \uparrow \\
A_P & & \xrightarrow{\quad \omega_{P|_{A_P}} \quad} & & \mathbb{V}_P \cong \mathbb{P}^{d-4}
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$$

Theorem (Irving, Schenck)

Let P be a d-gon in $\mathbb{P}^2$.

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\uparrow & & \uparrow & & \\
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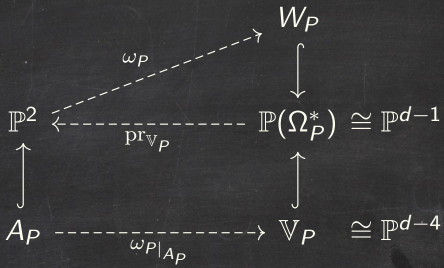
Wachspress Surfaces



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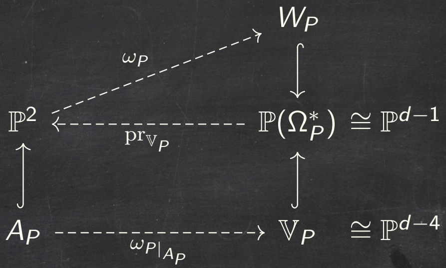
Wachspress Surfaces



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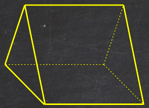
- *If $d = 4$, the image of the adjoint line A_P is a point.*

Wachspress Threefolds

P



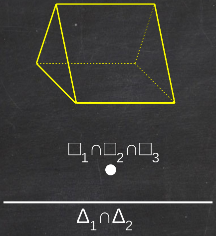
$$\frac{\square_1 \cap \square_2 \cap \square_3 \;\bullet}{\Delta_1 \cap \Delta_2}$$

$\mathcal{R}_P$

A_P adjoint plane

Wachspress Threefolds

P



$\mathcal{R}_P$

A_P adjoint plane

ω_P $(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$

Wachspress Threefolds

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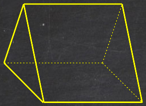

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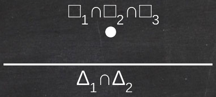


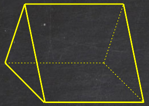
A_P adjoint plane

ω_P $(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$

W_P $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

Wachspress Threefolds

P



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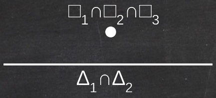

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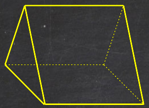
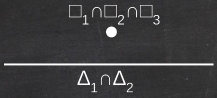
$\omega_{P|_{A_P}}$ projection from point

Wachspress Threefolds

P	
$\mathcal{R}_P$	
A_P	adjoint plane
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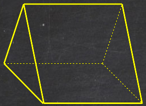
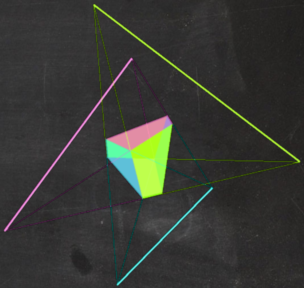
$\omega_{P\|_{A_P}}$	projection from point
$\omega_P(A_P)$	line

Wachspress Threefolds

P		
$\mathcal{R}_P$	 	
A_P	adjoint plane	adjoint quadric surface
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$	
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	
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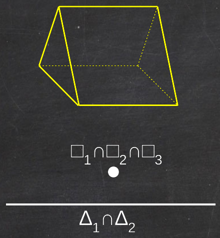
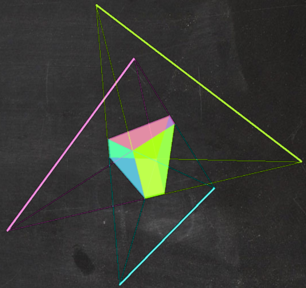
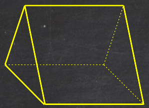
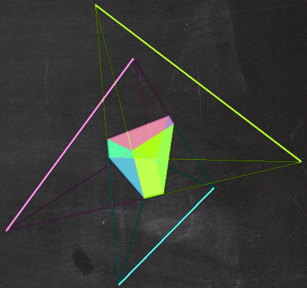
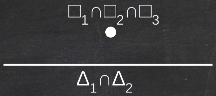
$\omega_P(A_P)$	line	

Wachspress Threefolds

P		
$\mathcal{R}_P$		
A_P	adjoint plane	adjoint quadric surface
ω_P	$(\ell_{\triangle_1} : \ell_{\triangle_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3})$	$(\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4)$
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	
$\omega_{P\|_{A_P}}$	projection from point	
$\omega_P(A_P)$	line	

Wachspress Threefolds

P		
$\mathcal{R}_P$		
A_P	adjoint plane	adjoint quadric surface
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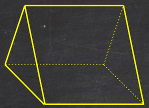
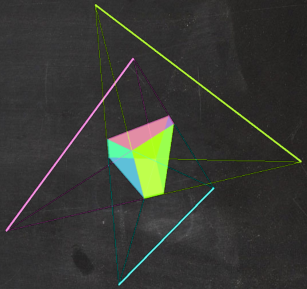
W_P	$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$
$\omega_{P\|_{A_P}}$	projection from point	
$\omega_P(A_P)$	line	

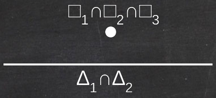
Wachspress Threefolds

P





$\mathcal{R}_P$



A_P	adjoint plane	adjoint quadric surface
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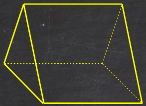
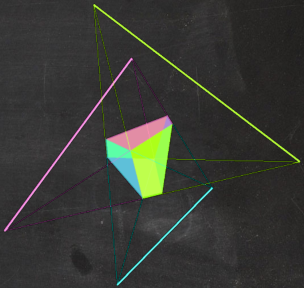
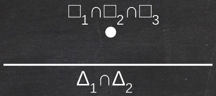
$\omega_{P\|_{A_P}}$	projection from point	contracts ruling of lines
$\omega_P(A_P)$	line	

Wachspress Threefolds

P		
$\mathcal{R}_P$		
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$\omega_{P\|_{A_P}}$	projection from point	contracts ruling of lines
$\omega_P(A_P)$	line	twisted cubic curve

Wachspress Threefolds

- P: polytope in $\mathbb{P}^3$ with d facets
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- a: number of isolated points in $\mathcal{R}_P$
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Proposition (K., Ranestad)

The Wachspress variety $W_P \subset \mathbb{P}^{2d-5}$ is a threefold of degree

$$2b + 4c - a - \frac{1}{2}(d-3)(d^2 - 11d + 26) = b + 2c + 1 - \frac{1}{6}(d-3)(d-4)(d-11)$$

and sectional genus $b + 2c + 1 + \frac{1}{2}(d-3)(d-6)$.

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$$2b+4c-a-\frac{1}{2}(d-3)(d-4)(d-6) = b+2c+1-\frac{1}{6}(d-3)(d^2-12d+38)$$

and its sectional genus is $b+2c+1-\frac{1}{2}(d-3)(d-4)$.

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
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Idea:

$$P \rightsquigarrow \mathcal{H}_P$$

hypersurface
of degree d

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polytopal hypersurface:
hypersurface of degree d,
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Idea:

$$
\begin{array}{ccccccc}
 & & & \mathbb{P}^n & \xleftarrow{\text{blowup } \pi} & X & \text{smooth} \\
 & & & \uparrow & & \uparrow & \\
P & \rightsquigarrow & \mathcal{H}_P & \rightsquigarrow & D & \rightsquigarrow & \tilde{D} \quad \text{smooth}
\end{array}
$$

Below $\mathcal{H}_P$: hypersurface of degree d

Below D: **polytopal hypersurface:** hypersurface of degree d, multiplicity c along $\mathcal{R}_P^c$, smooth outside of $\mathcal{R}_P$

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Idea:

$$\begin{array}{ccccccc} & & & & \mathbb{P}^n & \xleftarrow{\text{blowup } \pi} & X \quad \text{smooth} \\ & & & & \uparrow & & \uparrow \\ P & \rightsquigarrow & \mathcal{H}_P & \rightsquigarrow & D & \rightsquigarrow & \tilde{D} \quad \text{smooth} \end{array}$$

$\mathcal{H}_P$: hypersurface of degree d

D: **polytopal hypersurface:** hypersurface of degree d, multiplicity c along $\mathcal{R}_P^c$, smooth outside of $\mathcal{R}_P$

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

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polytopal hypersurface:
hypersurface of degree d,
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Idea:

$\mathbb{P}^n \xleftarrow{\text{blowup } \pi} X$ smooth

$D \hookrightarrow \mathbb{P}^n$, $\tilde{D} \hookrightarrow X$

$P \rightsquigarrow \mathcal{H}_P \rightsquigarrow D \rightsquigarrow \tilde{D}$ smooth

$\mathcal{H}_P$: hypersurface of degree d

D: **polytopal hypersurface:** hypersurface of degree d, multiplicity c along $\mathcal{R}_P^c$, smooth outside of $\mathcal{R}_P$

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$\tilde{D}$ has a unique adjoint A in X, and thus a unique canonical divisor: $A \cap \tilde{D}$. Moreover, $\pi(A) = A_P$.

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Idea:

$$\begin{tikzcd} & & \mathbb{P}^n & X \arrow[l, "\text{blowup } \pi"'] & \text{smooth} \\ P \arrow[r, rightsquigarrow] & \mathcal{H}_P \arrow[r, rightsquigarrow] & D \arrow[u, hook] \arrow[r, rightsquigarrow] & \tilde{D} \arrow[u, hook] & \text{smooth} \end{tikzcd}$$

hypersurface
of degree d

polytopal hypersurface:
hypersurface of degree d,
multiplicity c along $\mathcal{R}_P^c$,
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When can we find
a polytopal hypersurface D?

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

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Polytopal Hypersurfaces

Proposition (K., Ranestad)

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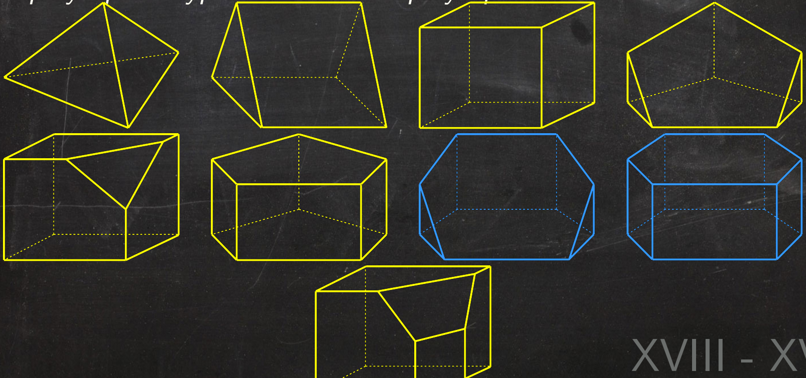
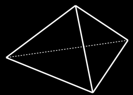
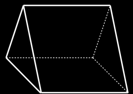
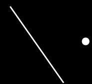
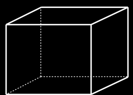
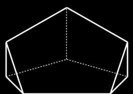
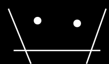
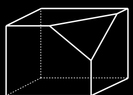
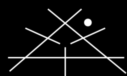
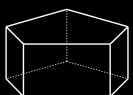
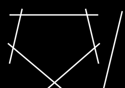
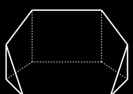
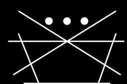
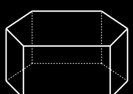
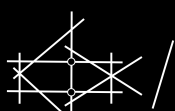
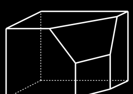
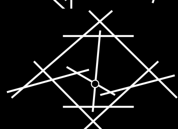
Theorem (K., Ranestad)

Let $\mathcal{C}$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let P be a general polytope of type $\mathcal{C}$. There is a polytopal surface D iff $\mathcal{C}$ is one of:



comb. type	facet sizes	$\mathcal{R}_P$	(a,b,c)	W_P (deg., sec. genus)	$\overline{w_P(A_P)}$ (deg., sec. genus)	$\dim \Gamma_P$	$\overline{w_P(D)}$ (deg., sec. genus)
	3 3 3 3		$(0,0,0)$	$\mathbb{P}^3$ $(1,0)$	0	34	minimal K3 (smooth quartic in $\mathbb{P}^3$)
	4 4 4 3 3		$(1,0,0)$	$\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ $(3,0)$	line	23	minimal K3 $(8,5)$
	4 4 4 4 4 4		$(0,0,0)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ $(6,1)$	twisted cubic curve	26	minimal K3 $(12,7)$
	5 5 4 4 3 3		$(2,2,0)$	$W_P \subset \mathbb{P}^7$ $(8,3)$	quadric surface $(2,0)$	17	non-minimal K3 $(14,9)$
	5 5 5 4 4 4 3		$(1,6,0)$	$W_P \subset \mathbb{P}^9$ $(15,9)$	del Pezzo surface in $\mathbb{P}^5$ $(5,1)$	7	non-minimal K3 $(19,12)$
	5 5 4 4 4 4 4		$(0,5,0)$	Fano 3-fold in $\mathbb{P}^9$ $(14,8)$	rational scroll in $\mathbb{P}^5$ $(4,0)$	12	non-minimal K3 $(18,11)$
	6 6 4 4 4 3 3		$(3,6,1)$	$W_P \subset \mathbb{P}^9$ $(17,11)$	rational elliptic surface in $\mathbb{P}^5$ $(7,3)$	4	minimal elliptic $(22,15)$
	6 6 4 4 4 4 4 4		$(0,12,2)$	$W_P \subset \mathbb{P}^{11}$ $(27,22)$	elliptic K3-surface in $\mathbb{P}^7$ $(12,7)$	3	minimal elliptic $(26,17)$
	5 5 5 5 4 4 4 4		$(0,16,0)$	$W_P \subset \mathbb{P}^{11}$ $(27,22)$	K3-surface in $\mathbb{P}^7$ $(12,7)$	1	non-minimal K3 $(24,15)$