Computing Chow forms, Hurwitz forms and Beyond

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Section 1

Chow forms

Chow hypersurfaces

Consider an irreducible variety X in $\mathbb{P}^n_{\mathbb{C}}$ of dimension k.

- A general linear space in $\mathbb{P}^n$ of dimension $n-k-1$ does not meet X.
- The set of such linear spaces meeting X is an irreducible hypersurface in the Grassmannian $\mathrm{Gr}(n-k-1, \mathbb{P}^n)$, called **Chow hypersurface** $\mathrm{Ch(X)}$.

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$\mathbb{P}^3$

curve C

$\mathrm{Ch}(C)$ lines meeting C

surface S

$\mathrm{Ch}(S) = S$ points meeting S

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curve C

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surface S

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threefold T

$\mathrm{Ch}(T) = T$ points meeting T

Chow forms

The Chow hypersurface $\mathrm{Ch}(X)$ determines X uniquely.

- $\mathrm{Ch}(X)$ is defined by 1 polynomial in Plücker coordinates, which is unique up to scaling and Plücker relations, called **Chow form** of X.
- The degree of the Chow form of X is $\deg(X)$.
- Chow forms of degree d in $\mathbb{C}[\mathrm{Gr}(n-k-1,\mathbb{P}^n)]$ parameterize k-dimensional subvarieties of $\mathbb{P}^n$ with degree d.

History:

- 1860: Cayley introduced these forms for space curves
- 1937: Chow and van der Waerden define them for arbitrary varieties

Section 2

Hurwitz forms and beyond

Hurwitz hypersurfaces

Consider an irreducible variety X in $\mathbb{P}^n$ of dimension k and degree $d \geq 2$.

- A general linear space in $\mathbb{P}^n$ of dimension $n-k$ intersects X at d points transversely.
- The Zariski closure of the set of such linear spaces that intersect X non-transversely at some smooth point is an irreducible hypersurface in $\mathrm{Gr}(n-k, \mathbb{P}^n)$, called **Hurwitz hypersurface** $\mathrm{Hur}(X)$. [Sturmfels]

Hurwitz hypersurfaces

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$\mathrm{Hur}(C) = C^\vee$	planes tangent to C

surface S

$\mathrm{Ch}(S) = S$	points meeting S
$\mathrm{Hur}(S)$	lines tangent to S

$\mathbb{P}^4$

curve C

$\mathrm{Ch}(C)$	planes meeting C
$\mathrm{Hur}(C) = C^\vee$	tangent hyperplanes

surface S

$\mathrm{Ch}(S)$	lines meeting S
$\mathrm{Hur}(S)$	planes containing tangent lines

threefold T

$\mathrm{Ch}(T) = T$	points meeting T
$\mathrm{Hur}(T)$	tangent lines

Coisotropic hypersurfaces

Consider an irreducible variety X in $\mathbb{P}^n$ of dimension k, and let $0 \leq i \leq k$.

- A general linear space in $\mathbb{P}^n$ of dimension $n - k + i$ intersects X transversely at all smooth intersection points.
- The Zariski closure of the set of such linear spaces that intersect X non-transversely at some smooth point is an irreducible variety in $\mathrm{Gr}(n - k + i, \mathbb{P}^n)$, called ***i*-th coisotropic variety** $\mathrm{CH}_i(X)$. [GKZ]

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- $\mathrm{CH}_i(X)$ is a hypersurface if and only if $i \leq k - \mathrm{codim}(X^\vee) + 1$. In this case, its defining equation is called ***i*-th coisotropic form** of X.

The dual variety $X^\vee$ is the Zariski closure in $(\mathbb{P}^n)^*$ of the set of all hyperplanes that are tangent to X at some smooth point.

Coisotropic hypersurfaces

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$\mathrm{Ch}(C)$ lines meeting C

$\mathrm{Hur}(C) = C^\vee$ planes tangent to C

surface S

$\mathrm{Ch}(S) = S$ points meeting S

$\mathrm{Hur}(S)$ lines tangent to S

$\mathrm{CH}_2(S) = S^\vee$ planes tangent to S

$\mathbb{P}^4$

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$\mathrm{Ch}(C)$ planes meeting C

$\mathrm{Hur}(C) = C^\vee$ tangent hyperplanes

surface S

$\mathrm{Ch}(S)$ lines meeting S

$\mathrm{Hur}(S)$ planes containing tangent lines

$\mathrm{CH}_2(S) = S^\vee$ tangent hyperplanes

threefold T

$\mathrm{Ch}(T) = T$ points meeting T

$\mathrm{Hur}(T)$ tangent lines

$\mathrm{CH}_2(T)$ tangent planes

$\mathrm{CH}_3(T) = T^\vee$ tangent hyperplanes

Applications

- Bürgisser, Lerario (2016): Unitary group acts transitively on the tangent spaces of a coisotropic hypersurface
  $\rightsquigarrow$ can compute volume of coisotropic hypersurfaces via kinematic formula
  $\rightsquigarrow$ **probabilistic Schubert calculus**
  $\rightsquigarrow$ Bürgisser (2015): **condition** of intersecting varieties with linear spaces

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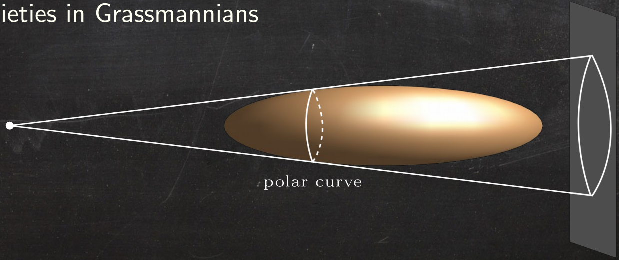
  $\rightsquigarrow$ Bürgisser (2015): **condition** of intersecting varieties with linear spaces
- **hyperdeterminants** are coisotropic forms in matrix coordinates
- K., Sturmfels, Trager (2017): The (iterated) singular loci of the coisotropic hypersurfaces of a space curve or surface X parameterize the **visual events** of X
- Connections to **polar geometry**: Coisotropic hypersurfaces are the analogue of polar varieties in Grassmannians



Computational questions

Every coisotropic hypersurface $\mathrm{CH}_i(X)$ determines X uniquely.

- How to compute the i-th coisotropic form of X from $\mathrm{I}(X)$?
- How to compute $\mathrm{I}(X)$ and i from a coisotropic form?
- How to test if a polynomial in Plücker coordinates is a coisotropic form?

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General algorithms that compute coisotropic forms:

- Input: any homogeneous ideal I, any index i
- Output: i-th coisotropic form of $V(\mathrm{I})$
- `Macaulay2` package "Resultants"
- `Macaulay2` package "Coisotropy" (only available from my website)
- these general implementations are slow
  $\rightsquigarrow$ develop specialized algorithms for computable cases

Test coisotropy and recover (X, i)

- Let Q be a polynomial in Plücker coordinates of $\mathrm{Gr}(k, \mathbb{P}^n)$.
- Every k-dimensional linear space in $\mathbb{P}^n$ is the kernel of an $(n-k) \times (n+1)$-matrix $A = (a_{i,j})$.
- Substitute Plücker coordinates by maximal minors of A to derive $\tilde{Q} \in \mathbb{C}[a_{i,j}]$ from Q.
- Form the $(n-k) \times (n+1)$-Jacobian matrix $J = (\frac{\partial \tilde{Q}}{\partial a_{i,j}})$.

Q is a coisotropic form if and only if $\mathrm{rank}(J(A)) \leq 1$ for all $A \in V(\tilde{Q})$.

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In this case, $V(\tilde{Q})^\vee = \mathbb{P}^{n-k-1} \times X$ and $n-k-1 = \dim(X) - i$.

$$
\begin{aligned}
&16p_{01}^4p_{03}^4p_{13}^4 - 36p_{01}^2p_{03}^6p_{13}^4 - 16p_{02}p_{03}^7p_{13}^4 + \\
&96p_{01}^4p_{03}^3p_{13}^5 - 324p_{01}^2p_{03}^5p_{13}^5 - 192p_{02}p_{03}^6p_{13}^5 + \\
&256p_{01}^6p_{13}^6 - 504p_{01}^4p_{03}^2p_{13}^6 - 320p_{01}^2p_{02}p_{03}^3p_{13}^6 - \\
&972p_{01}^2p_{03}^4p_{13}^6 - 864p_{02}p_{03}^5p_{13}^6 - 1944p_{01}^4p_{03}p_{13}^7 - \\
&1920p_{01}^2p_{02}p_{03}^2p_{13}^7 - 972p_{01}^2p_{03}^3p_{13}^7 - \\
&1728p_{02}p_{03}^4p_{13}^7 - 768p_{01}^4p_{12}p_{13}^7 - 2187p_{01}^4p_{13}^8 - \\
&4176p_{01}^2p_{02}p_{03}p_{13}^8 - 128p_{02}^2p_{03}^2p_{13}^8 - \\
&1296p_{02}p_{03}^3p_{13}^8 + 768p_{01}^2p_{12}^2p_{13}^8 - 3888p_{01}^2p_{02}p_{13}^9 - \\
&768p_{02}^2p_{03}p_{13}^9 - 256p_{12}^3p_{13}^9 - 1152p_{02}^2p_{13}^1 0 + \\
&24p_{01}p_{03}^6p_{13}^4p_{23} + 32p_{01}^3p_{03}^3p_{13}^5p_{23} + \\
&216p_{01}p_{03}^5p_{13}^5p_{23} + 624p_{01}^3p_{03}^2p_{13}^6p_{23} + \\
&648p_{01}p_{03}^4p_{13}^6p_{23} + 936p_{01}^3p_{03}p_{13}^7p_{23} - \\
&320p_{01}p_{02}p_{03}^2p_{13}^7p_{23} + 648p_{01}p_{03}^3p_{13}^7p_{23} + \\
&972p_{01}^3p_{13}^8p_{23} + 1056p_{01}p_{02}p_{03}p_{13}^8p_{23} - \\
&288p_{01}p_{02}p_{13}^9p_{23} - 4p_{03}^6p_{13}^4p_{23}^2 - 36p_{03}^5p_{13}^5p_{23}^2 - \\
&56p_{01}^2p_{03}^2p_{13}^6p_{23}^2 - 108p_{03}^4p_{13}^6p_{23}^2 + \\
&312p_{01}^2p_{03}p_{13}^7p_{23}^2 - 108p_{03}^3p_{13}^7p_{23}^2 - 18p_{01}^2p_{13}^8p_{23}^2 - \\
&144p_{02}p_{03}p_{13}^8p_{23}^2 - 432p_{02}p_{13}^9p_{23}^2 - \\
&72p_{01}p_{03}p_{13}^7p_{23}^3 + 108p_{01}p_{13}^8p_{23}^3 - 27p_{13}^8p_{23}^4
\end{aligned}
$$

Thanks for your attention