Coisotropic Hypersurfaces in Algebraic Vision

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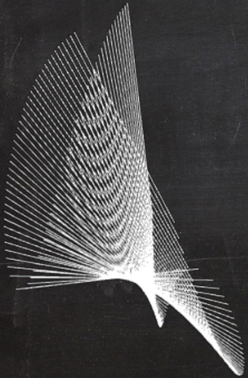


Section 1

Preliminaries

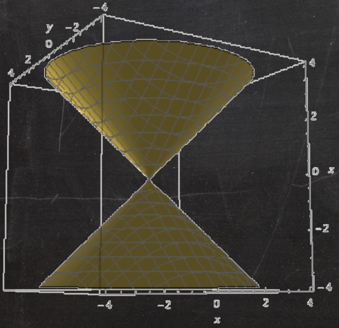
Projective Varieties

Projective space $\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where

$$x \sim y \;\Leftrightarrow\; \exists \lambda \in \mathbb{C} \setminus \{0\} : x = \lambda y$$

A **projective variety** $X \subset \mathbb{P}^n$ is the 0-set of homogeneous polynomials in $n+1$ variables

Example: circle in projective plane, defined by $x^2 + y^2 - z^2$



Grassmannians

$$\mathrm{Gr}(k, \mathbb{C}^n) := \{L \subset \mathbb{C}^n \mid L \text{ is a } k\text{-dimensional subspace}\}$$
$$= \mathrm{Gr}(k-1, \mathbb{P}^{n-1})$$

is a projective variety

Example: $\mathrm{Gr}(2, \mathbb{C}^4) = \mathrm{Gr}(1, \mathbb{P}^3) = \{\text{lines in } \mathbb{P}^3\}$

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Let $L \in \mathrm{Gr}(1, \mathbb{P}^3)$ be spanned by rows of $\left(\begin{smallmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{smallmatrix}\right)$
$\Rightarrow$ For $i < j$, let p_{ij} be minor of $\left(\begin{smallmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{smallmatrix}\right)$ using columns i, j
$\Rightarrow p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$

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$\Rightarrow p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$
This even gives an embedding

$$\mathrm{Gr}(1, \mathbb{P}^3) \hookrightarrow \mathbb{P}^5,$$
$$L \longmapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$$

$\mathrm{Gr}(1, \mathbb{P}^3)$ is a hypersurface in $\mathbb{P}^5$ defined by $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$

Section 2

Coisotropic Hypersurfaces

Coisotropic Hypersurfaces

A **coisotropic hypersurface** in $\mathrm{Gr}(k, \mathbb{P}^n)$ consists of those $L \in \mathrm{Gr}(k, \mathbb{P}^n)$ that intersect a given variety non-transversally

Example: Let $C \subset \mathbb{P}^3$ be a curve

all lines intersecting C

form a hypersurface in $\mathrm{Gr}(1, \mathbb{P}^3)$

all planes tangent to C

form a hypersurface in $\mathrm{Gr}(2, \mathbb{P}^3)$

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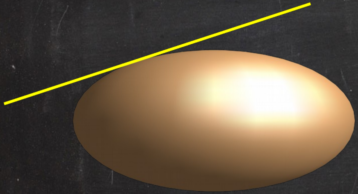
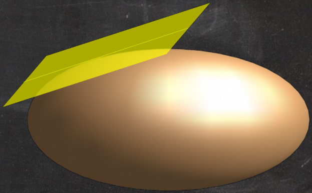
Example: Let $S \subset \mathbb{P}^3$ be a surface



all lines tangent to S

form a hypersurface in $\mathrm{Gr}(1, \mathbb{P}^3)$



all planes tangent to S

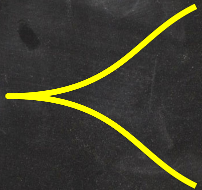
form a hypersurface in $\mathrm{Gr}(2, \mathbb{P}^3)$

Singular Points

A point x on a variety X is **singular** if X does not look like a manifold locally around x

Example: some singularities on plane curves

node



cusp

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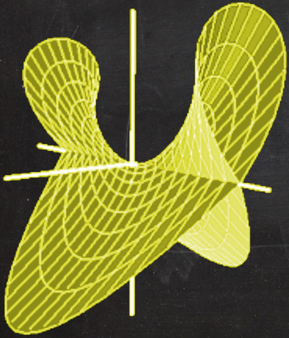
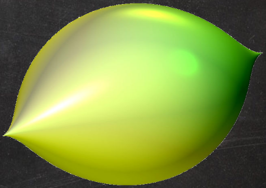
Example: some singularities on surfaces





Singular Points

A point x on a variety X is **singular** if X does not look like a manifold locally around x

Example: coisotropic hypersurfaces of a curve $C \subset \mathbb{P}^3$

all lines intersecting C

singular points:
lines intersecting C twice

all planes tangent to C

singular points:

planes tangent to C twice

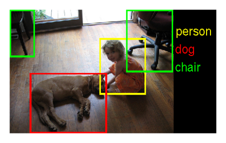
planes intersecting C with contact order 3

Section 3

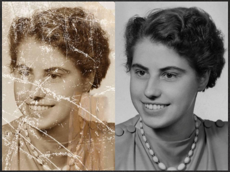
Algebraic Vision

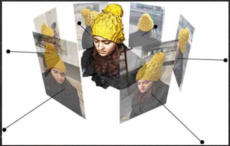
Algebraic Vision

= Algebraic Geometry + Computer Vision



- object recognition
- image restoration
- 3D scene reconstruction
- event detection
- etc.

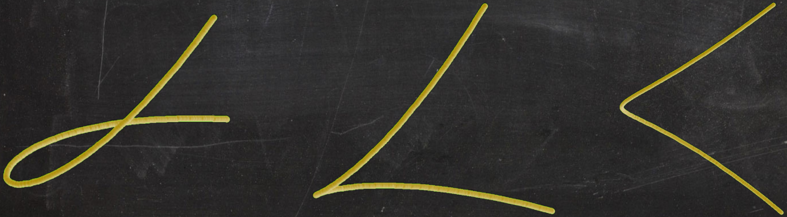




Event Detection on Curves

Take pictures of a 3D curve with a moving camera

The 2D pictures from general camera points contain only nodes



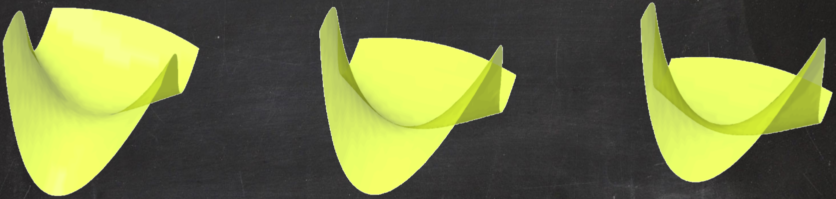
If the camera point lies on a tangent line of the curve, the picture has a cusp

Event Detection on Curves

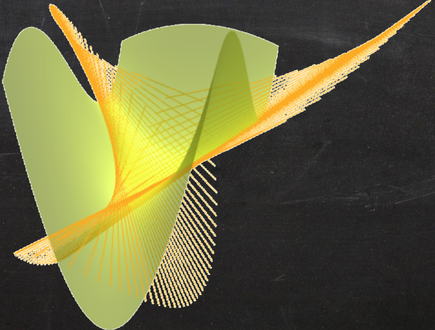
Tangent Developable

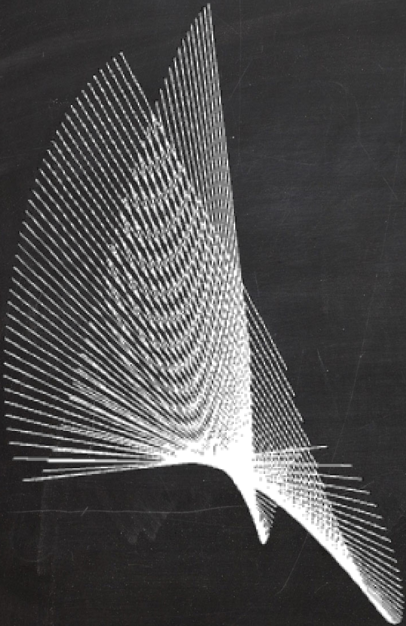
Event Detection on Surfaces

Take pictures of a surface with a moving camera



All camera lines where the situation in the middle happens:





Thanks for your attention