NCG Group
First Results and Open Problems

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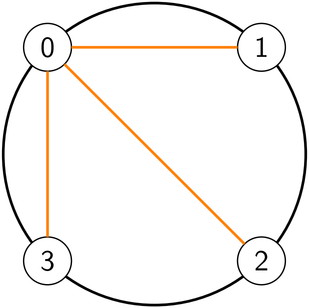
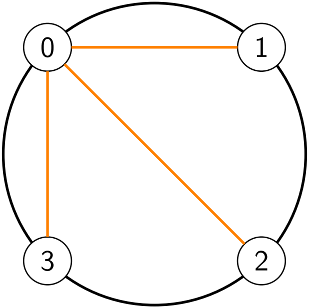
Node Swap Problem

Non-existence of Nash equilibria



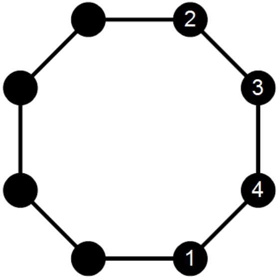
Non-existence of Nash equilibria



Lemma (Non-existence of NE)

For any connected graph $G = (V, E)$ with diameter at least two, there is a reachable friendship situation, for which no equilibrium exists.

No convergence



- Nodes on static ring topology
- No initial friendships, friends get externally added whenever an equilibrium is reached
- **Nodes can swap position with a neighbor**
- $v \in V$ minimizes $\max_{u \in \mathcal{F}(v)} d(u, v)$

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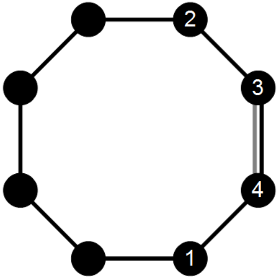

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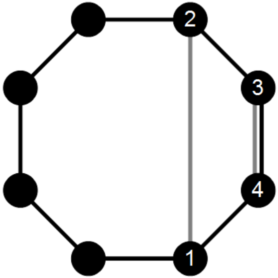

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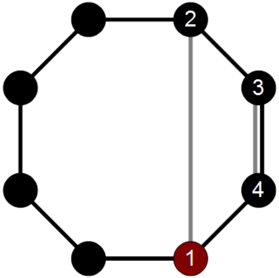

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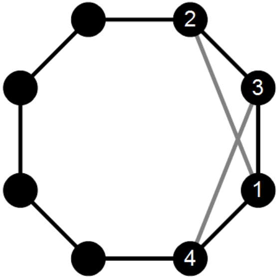

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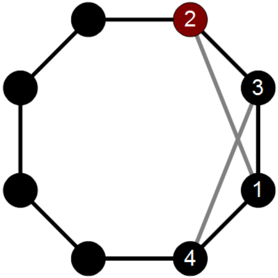

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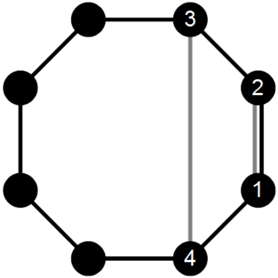

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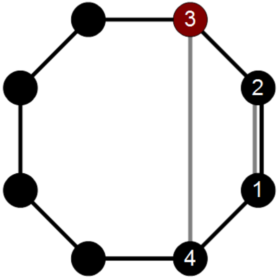

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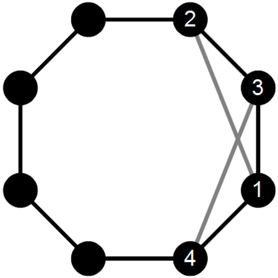

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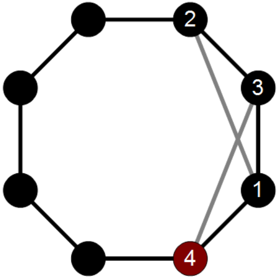

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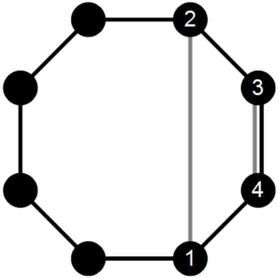

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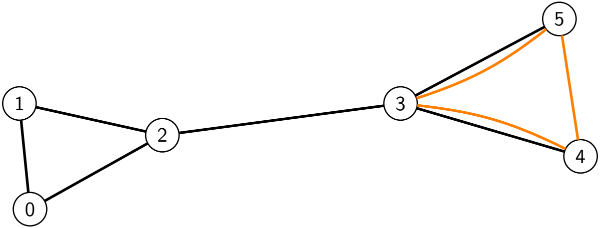
No convergence

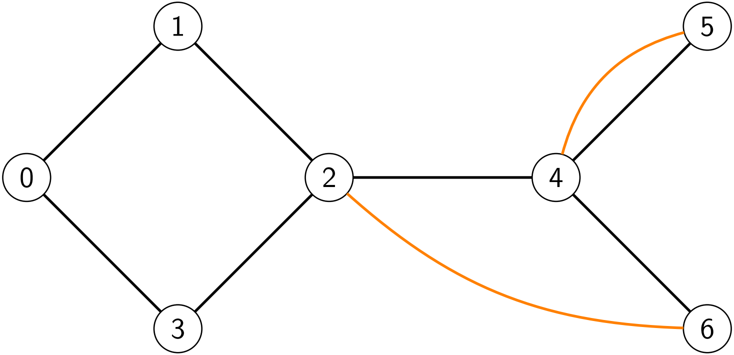
Lemma (No convergence)

For any connected graph $G = (V, E)$ with diameter at least two, there is a reachable friendship situation, such that an equilibrium cannot be reached with an arbitrary sequence of node turns, even if it exists.

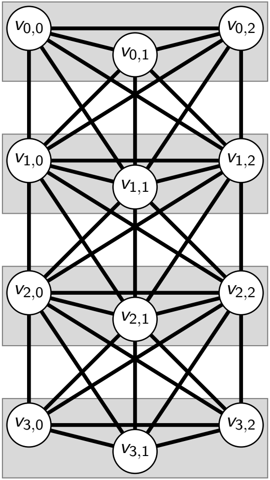


Convergence if friendships are matching?

Counterexample:

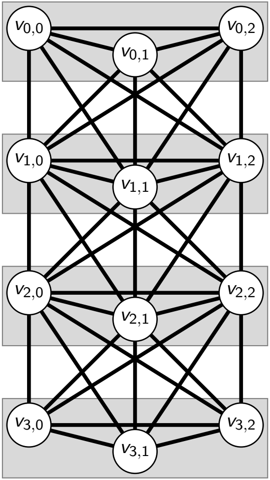


Convergence if friendships are matching?



complete layered graph

Convergence if friendships are matching?



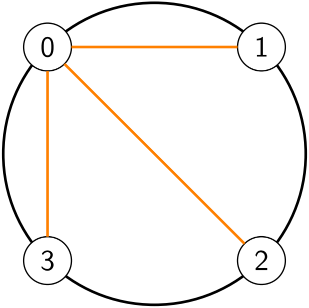
complete layered graph

Lemma

For any friendships that build a matching, there is a sequence of improving node switches such that a Nash equilibrium is reached.

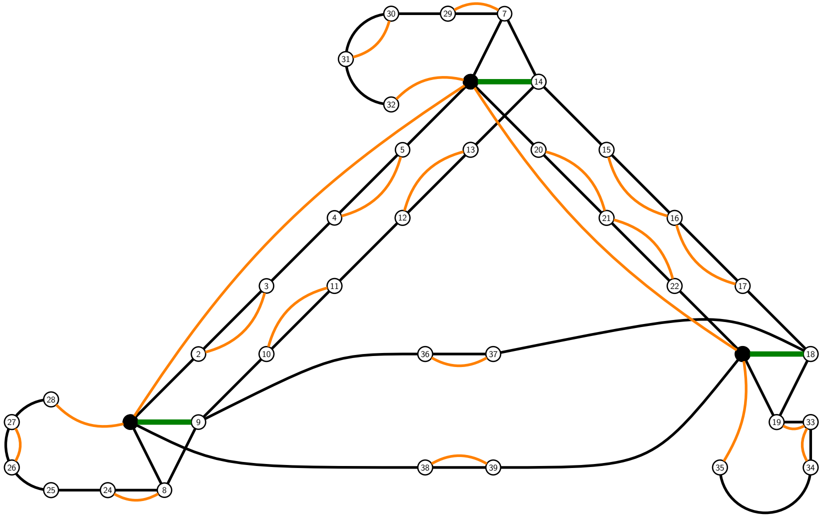
Pairwise Node Swap Problem

Pairwise Node Swap Problem



Nodes only swap if partner agrees.
$\Rightarrow$ Picture shows Nash equilibrium

Maximum Distance: No convergence



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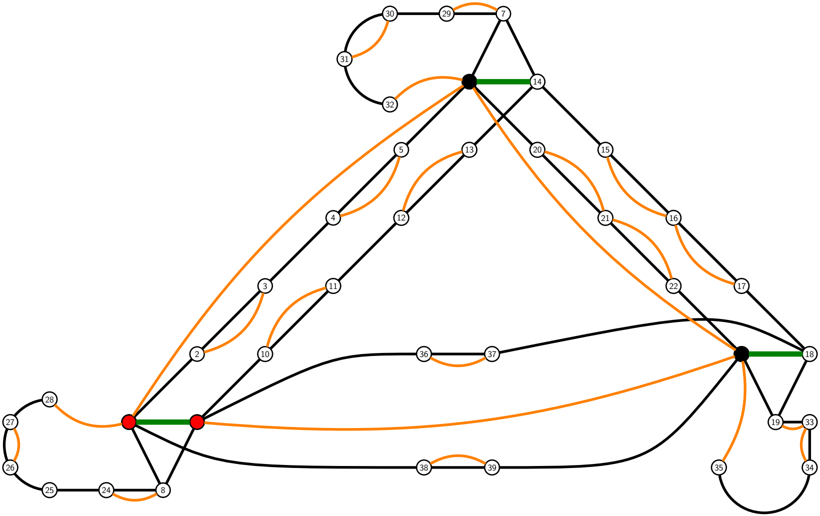

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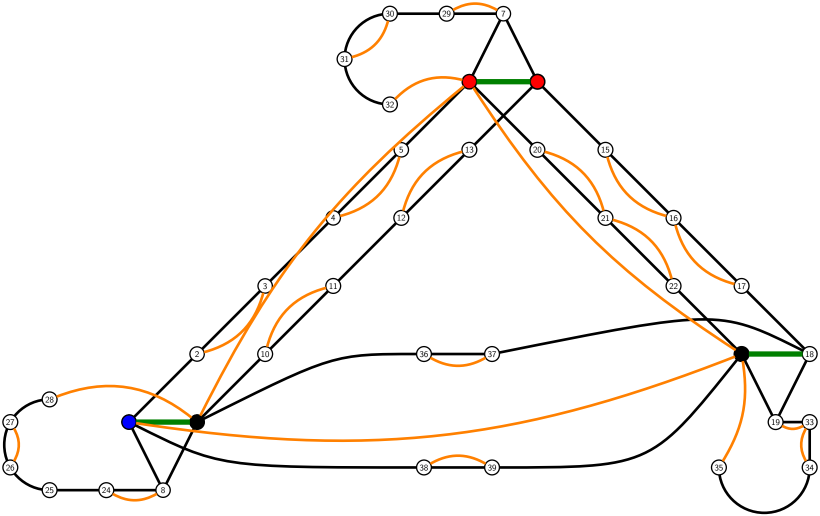

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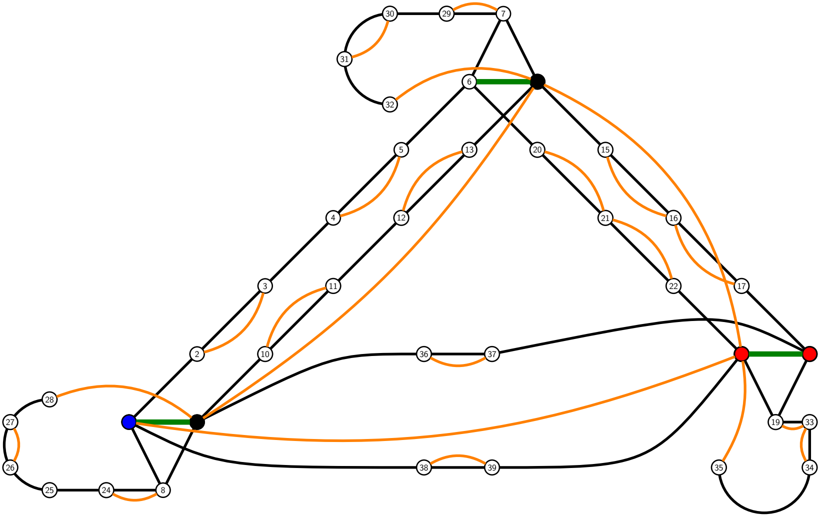

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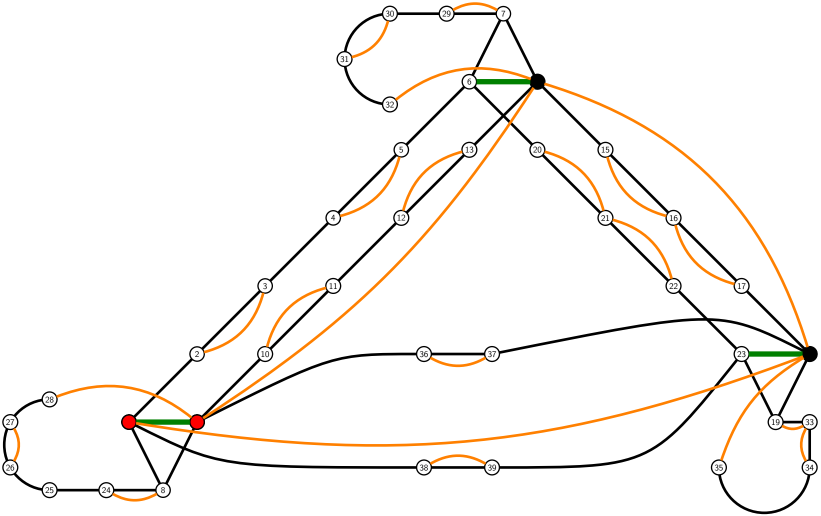


Average Distance: Always convergence to NE! $*$weeeeee$*$

Lemma

For every instance of the PNSP with an arbitrary sequence of improving moves and the sum of all distances to friends as metric, an equilibrium exists and the game converges.

Average Distance: Always convergence to NE! ∗weeeeee∗

Lemma

For every instance of the PNSP with an arbitrary sequence of improving moves and the sum of all distances to friends as metric, an equilibrium exists and the game converges.

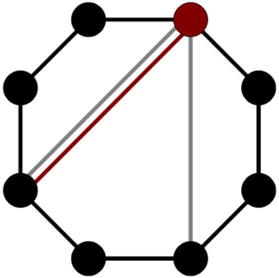
Proof.

- $f_G : V(G) \to \mathbb{N}_0, v \mapsto \sum_{u \in F(v)} \mathrm{d}_G(u, v)$
- Potential function $\Phi : \mathcal{G} \to \mathbb{N}_0, G \mapsto \sum_{v \in V(G)} f_G(v)$

□

Shortcut Problem

Shortcut Problem



- Nodes on static ring topology
- No initial friendships, friends get externally added whenever an equilibrium is reached
- Nodes can maintain up to 1 additional edge (without costs)
- $v \in V$ minimizes $\max_{u \in \mathcal{F}(v)} d(u, v)$

Trivial on diameter-2-graph

Lemma

Let $G = (V, E)$ be a graph with diameter 2 and all nodes are allowed to maintain up to a constant number $c \in \mathbb{N}$ of shortcuts. If a new friendship is added to a Nash equilibrium, the next Nash equilibrium is reached after at most one improving move. (This result is independent of the objective function: maximum or average.)

Open problems and discussion

Open Problems

1. PNSP: Existence of NE under maximum distance
2. Shortcut: Any results on
    - Complete layered graph
    - double-star / tree with diameter 3

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