NCG Group
New Results and Open Problems

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NP-Completeness

(P)NSP$_{\mathrm{OPT}}$ - Problem Definition

- An instance $\mathcal{I}$ of $(P)NSP_{opt}$: $\mathcal{I} = (G, F)$, where $G = (V, E)$ is a graph and F is a set of friendships. $n := |V|$
- Metric costs $f_G(v)$ of $v \in V$ can be $\max_{u \in F(v)} d_G(u, v)$ or $\frac{1}{|F(v)|} \sum_{u \in F(v)} d_G(u, v)$
- $\sum_{v \in V} f_G(v)$ has to be minimized in an Nash Equilibrium (NE)
- An instance $\mathcal{I}$ of $(P)NSP_{dec}$: $\mathcal{I} = (G, k, F)$, whereat G, F and n are the same as for $(P)NSP_{opt}$. The question: $\sum_{v \in V} f_G(v) \leq k$ possible?

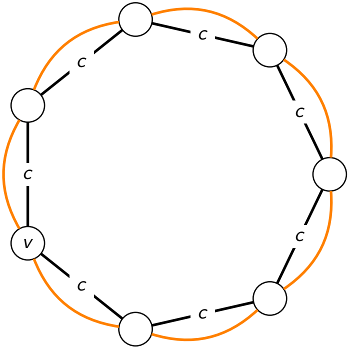
(P)NSP$_{\text{OPT}}$ - Reduction

- What we show: $CLIQUE_{dec} \preceq_p NSP_{dec}$
- Reduction function: $r((G, k)) := (G, k, F)$,
- whereat F is a set of friendships which precisely guarantees k pairwise friended nodes.
- $\mathcal{I} \in CLIQUE_{dec} \Leftrightarrow r(\mathcal{I}) \in NSP_{dec}$

SC_{OPT} - Reduction

- Metric costs $f_G(v)$ of $v \in V$ can be $\max_{u \in F(v)} d_G(u, v)$ or $\sum_{u \in F(v)} d_G(u, v)$
- SC_{opt} (SC_{dec}) analogously to $(P)NSP_{opt}$ ($(P)NSP_{dec}$)
- For a boolean $3 - CNF$ function ψ: $|\psi|_C$ is the number of clauses and $|\psi|_A$ is the number of atoms
- What we show: $3 - CNF - SAT \preceq_p SC_{dec}$
- Reduction function: $r(\psi) := (G, F, 18 \cdot |\psi|_C + 26 \cdot |\psi|_A)$ or $r(\psi) := (G, F, 34 \cdot |\psi|_C + 54 \cdot |\psi|_A)$

SC_{OPT} - Graphical Reduction - Shortcut fixation



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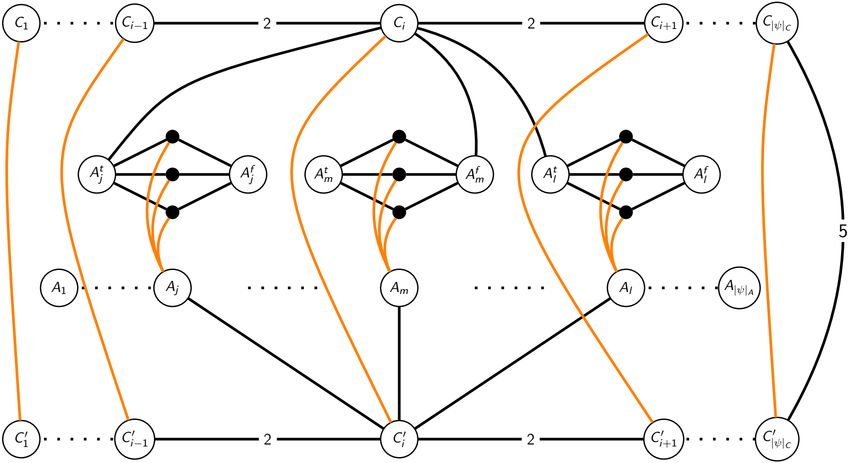
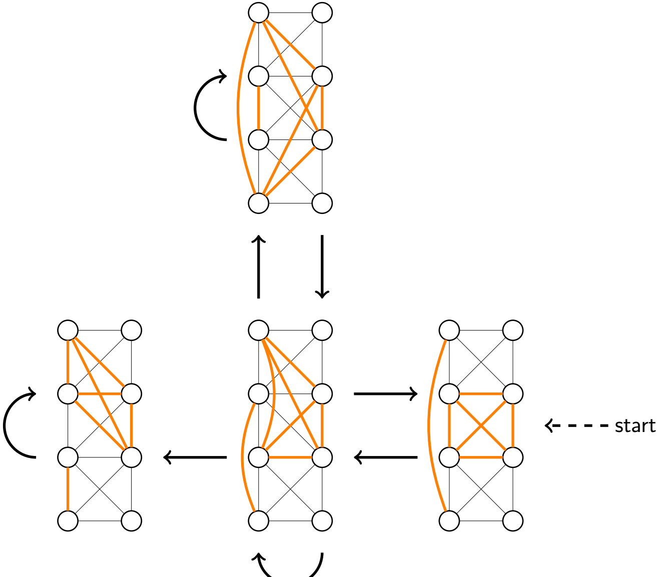
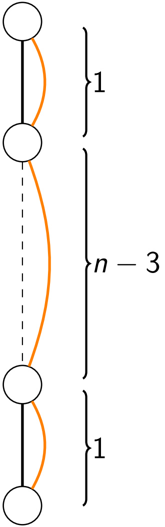


NSP on n-layered complete graphs

Friendships as subgraphs: NE reachable?

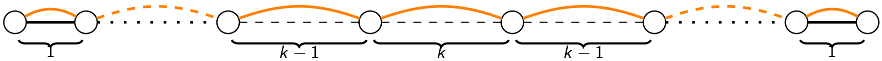


Average metric



1.
   - $n-3$ is maximum distance between friends
   - reachable friendship example $\Rightarrow$ tight
2.
   - worst-case PoA upper bounded by $n-3$
   - reachable friendship example $\Rightarrow \mathrm{PoA} = \Theta(n)$

Maximum metric



1.
   - $\lfloor\sqrt{n-1}\rfloor$ is maximum distance between friends
   - (non-reachable) friendship example $\Rightarrow$ tight
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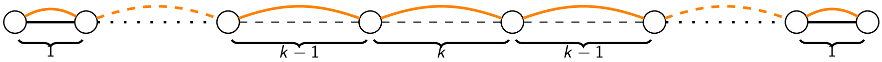
   - worst-case PoA upper bounded by $\lfloor\sqrt{n-1}\rfloor$
   - (non-reachable) friendship example $\Rightarrow \mathrm{PoA} = \Theta(\sqrt{n})$
3. Reachable friendship example for $\Theta(\sqrt{n})$ tightness (both statements)

Maximum metric



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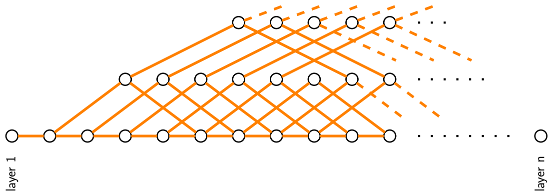

PNSP

Average metric
Convergence speed

- At most $|F|(\text{diam}(G) - 1)$ improving moves until next NE

- Non-reachable friendship example $\Rightarrow$ tight
- reachable friendship example $\Rightarrow$ tight in $\Theta(|F|(\text{diam}(G) - 1))$

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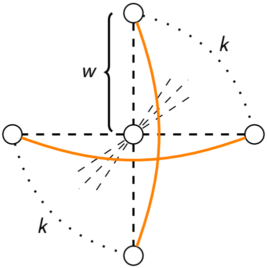

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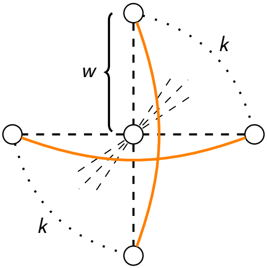

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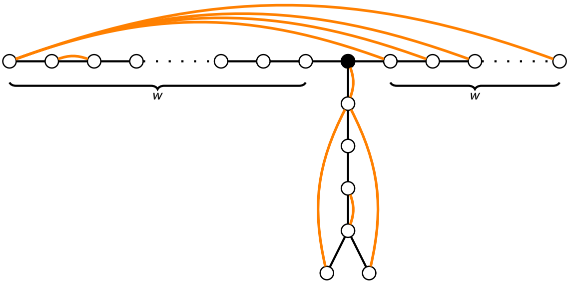

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PoA

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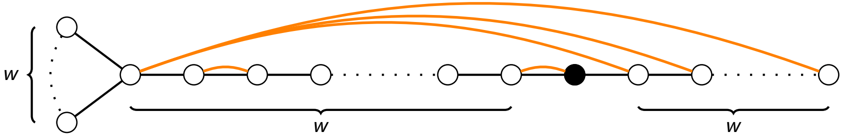

PNSP2: Maximum metric
Convergence

PNSP2: for swapping, both nodes need to improve

Lemma

For every instance of the PNSP2 with an arbitrary sequence of improving moves and the maximum of all distances to friends as metric, an equilibrium exists and the game converges.

Proof.

Potential function

$$\Phi : \mathcal{G} \to \mathbb{N}_0^*, G \mapsto \mathrm{sort} \searrow \left(\max_{u \in F(v)} d_G(u, v) \mid v \in V(G) \right)$$ □

At most $\binom{|V_F| + \mathrm{diam}(G) - 1}{|V_F|}$ improving moves until next NE

$(V_F := \{v \in V \mid F(v) \neq \emptyset\})$

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PoA

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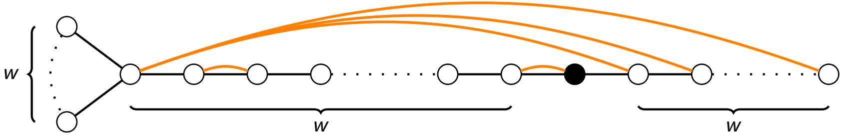

Open problems

- PoA for NSP in general graphs
- Tightness for convergence of PNSP2
- PoS
- Characterization of graphs/friendships for convergence/good NE
- Shortcut problem anyone?