NCG Group
New Results and Open Problems
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NP-Completeness

NSP on $n$-layered complete graphs

PNSP

Open problems
(P)NSP_{OPT} - Problem Definition

- An instance $\mathcal{I}$ of $(P)\text{NSP}_{opt}$: $\mathcal{I} = (G, F)$, where $G = (V, E)$ is a graph and $F$ is a set of friendships. $n := |V|$

- Metric costs $f_G(v)$ of $v \in V$ can be $\max_{u \in F(v)} d_G(u, v)$ or

$$\frac{1}{|F(v)|} \sum_{u \in F(v)} d_G(u, v)$$

- $\sum_{v \in V} f_G(v)$ has to be minimized in an Nash Equilibrium (NE)

- An instance $\mathcal{I}$ of $(P)\text{NSP}_{dec}$: $\mathcal{I} = (G, k, F)$, whereat $G$, $F$ and $n$ are the same as for $(P)\text{NSP}_{opt}$. The question: $\sum_{v \in V} f_G(v) \leq k$ possible?
What we show: $\text{CLIQUE}_{\text{dec}} \preceq_p \text{NSP}_{\text{dec}}$

Reduction function: $r((G, k)) := (G, k, F)$,

whereat $F$ is a set of friendships which precisely guarantees $k$ pairwise friended nodes.

$I \in \text{CLIQUE}_{\text{dec}} \iff r(I) \in \text{NSP}_{\text{dec}}$
**SC\textsubscript{OPT} - Reduction**

- **Metric costs** \( f_G(v) \) of \( v \in V \) can be \( \max_{u \in F(v)} d_G(u, v) \) or \( \sum_{u \in F(v)} d_G(u, v) \)

- \( SC_{opt} (SC_{dec}) \) analogously to \( (P)NSP_{opt} ((P)NSP_{dec}) \)

- For a boolean 3–CNF function \( \psi \): \( |\psi|_C \) is the number of clauses and \( |\psi|_A \) is the number of atoms

- What we show: 3–CNF–SAT \( \leq_p SC_{dec} \)

- Reduction function: \( r(\psi) := (G, F, 18 \cdot |\psi|_C + 26 \cdot |\psi|_A) \) or \( r(\psi) := (G, F, 34 \cdot |\psi|_C + 54 \cdot |\psi|_A) \)
SC_{OPT} - Graphical Reduction - Shortcut fixation
**SCOPT - Graphical Reduction**

Diagram showing a graph with nodes labeled as $C_1$, $C_{i-1}$, $C_i$, $C_{i+1}$, $C_{|\psi\rangle_C}$, $A_j^t$, $A_j^f$, $A_m^t$, $A_m^f$, $A_t^t$, $A_t^f$, $A_{1}$, ..., $A_{|\psi\rangle_A}$, etc. The diagram includes edges labeled with numerals "2" and "5".

Note: The specific details of the reduction process are not fully visible in the diagram due to the resolution and visibility limitations.
NSP on $n$-layered complete graphs
Friendships as subgraphs: NE reachable?
Average metric

$n - 3$ is maximum distance between friends
- reachable friendship example $\Rightarrow$ tight

worst-case PoA upper bounded by $n - 3$
- reachable friendship example
  $\Rightarrow$ PoA = $\Theta(n)$
Maximum metric

1. \([\sqrt{n} - 1]\) is maximum distance between friends
   - (non-reachable) friendship example \(\Rightarrow\) tight

2. worst-case PoA upper bounded by \([\sqrt{n} - 1]\)
   - (non-reachable) friendship example \(\Rightarrow\) PoA = \(\Theta(\sqrt{n})\)

3. Reachable friendship example for \(\Theta(\sqrt{n})\) tightness (both statements)
Maximum metric

1. $\lfloor \sqrt{n} - 1 \rfloor$ is maximum distance between friends
   (non-reachable) friendship example $\Rightarrow$ tight

2. worst-case PoA upper bounded by $\lfloor \sqrt{n} - 1 \rfloor$
   (non-reachable) friendship example $\Rightarrow$ PoA = $\Theta(\sqrt{n})$

3. Reachable friendship example for $\Theta(\sqrt{n})$ tightness (both statements)
NP-Completeness
NSP on $n$-layered complete graphs
PNSP
Open problems

PNSP
At most $|F|(\text{diam}(G) - 1)$ improving moves until next NE

- Non-reachable friendship example $\Rightarrow$ tight
- reachable friendship example $\Rightarrow$ tight in $\Theta(|F|(\text{diam}(G) - 1))$
Average metric
Convergence speed

- At most $|F|(\text{diam}(G) - 1)$ improving moves until next NE

- Non-reachable friendship example $\Rightarrow$ tight

- reachable friendship example $\Rightarrow$ tight in $\Theta(|F|(\text{diam}(G) - 1))$
Average metric
Convergence speed

- At most $|F|(\text{diam}(G) - 1)$ improving moves until next NE
- Non-reachable friendship example $\Rightarrow$ tight
- reachable friendship example $\Rightarrow$ tight in $\Theta(|F|(\text{diam}(G) - 1))$
Worst case PoA upper bounded by $\text{diam}(G)$

reachable friendship example $\Rightarrow$ tight in $\Theta(\text{diam}(G))$
Average metric
PoA

- Worst case PoA upper bounded by diam($G$)
- reachable friendship example $\Rightarrow$ tight in $\Theta$(diam($G$))
PNSP2: Maximum metric

Convergence

PNSP2: for swapping, both nodes need to improve

Lemma

For every instance of the PNSP2 with an arbitrary sequence of improving moves and the maximum of all distances to friends as metric, an equilibrium exists and the game converges.

Proof.

Potential function

\[ \Phi : G \rightarrow \mathbb{N}_0^*, \ G \mapsto \text{sort} \ \overset{\downarrow}{\text{max}} \left( \max_{u \in F(v)} d_G(u, v) \mid v \in V(G) \right) \]

At most \( \left( \frac{|V_F| + \text{diam}(G) - 1}{|V_F|} \right) \) improving moves until next NE

\( (V_F := \{ v \in V \mid F(v) \neq \emptyset \}) \)
PNSP2: for swapping, both nodes need to improve

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For every instance of the PNSP2 with an arbitrary sequence of improving moves and the maximum of all distances to friends as metric, an equilibrium exists and the game converges.

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At most \[\left( \frac{|V_F| + \text{diam}(G) - 1}{|V_F|} \right)\] improving moves until next NE

\((V_F := \{v \in V \mid F(v) \neq \emptyset\})\)
PNSP2: Maximum metric PoA

- Worst case PoA upper bounded by $\text{diam}(G)$
- Reachable friendship example $\Rightarrow$ tight in $\Theta(\text{diam}(G))$
Open problems
- PoA for NSP in general graphs
- Tightness for convergence of PNSP2
- PoS
- Characterization of graphs/friendships for convergence/good NE
- Shortcut problem anyone?