# The geometry of neural networks

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joint with

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$x \in \mathbb{R}^{d_x}$

$y \in \mathbb{R}^{d_y}$

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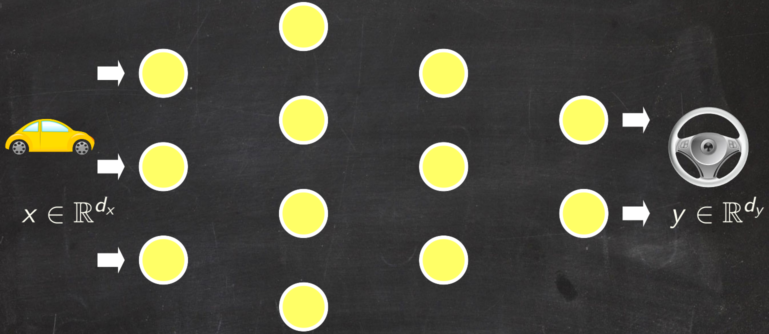

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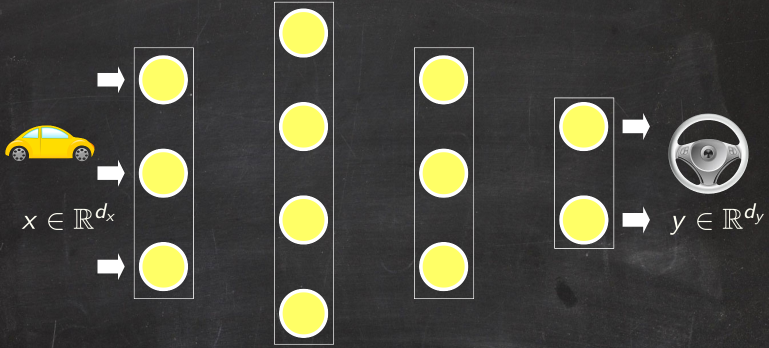

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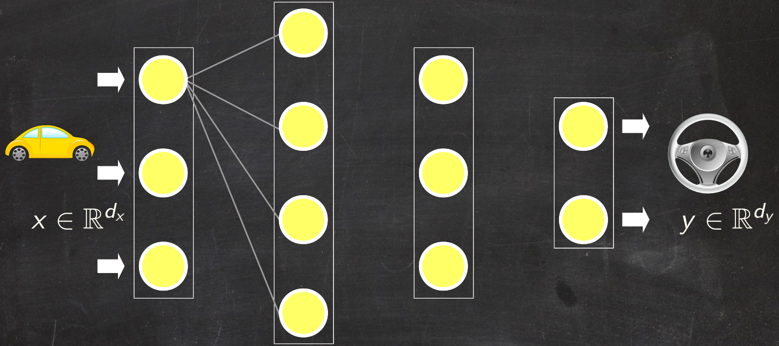

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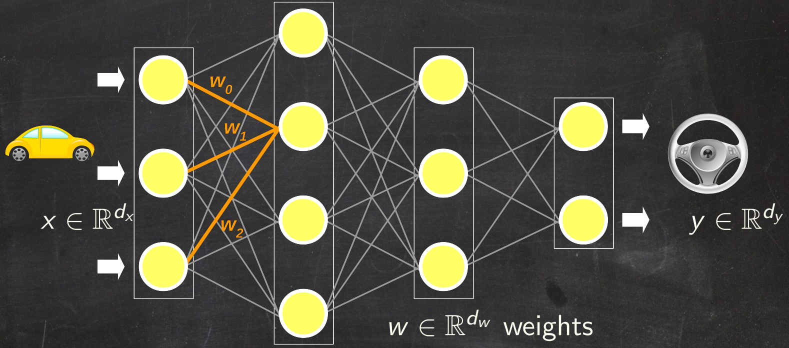

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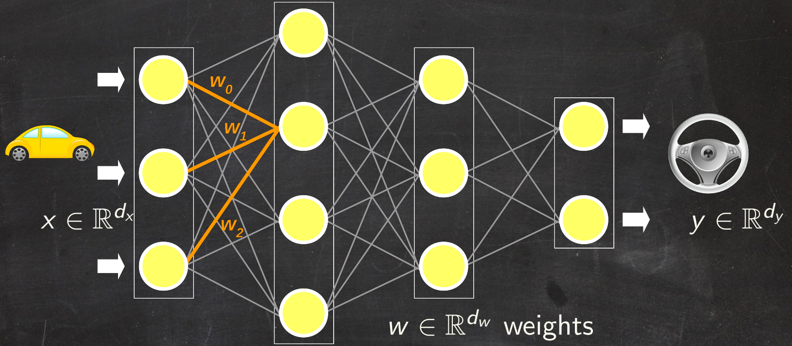


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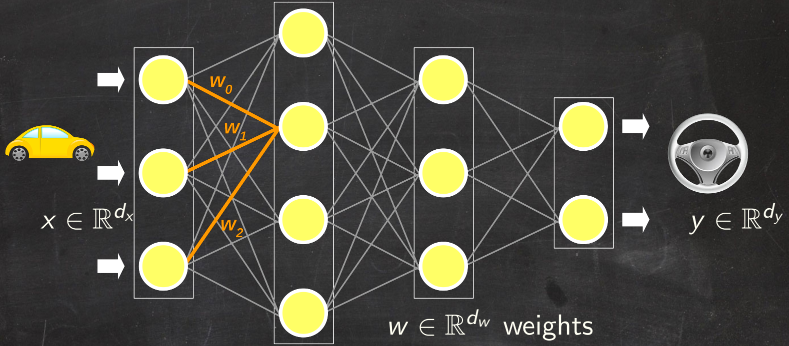


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**Definition** $\mathcal{M}_\Phi := \left\{ \Phi(w, \cdot) : \mathbb{R}^{d_x} \to \mathbb{R}^{d_y} \mid w \in \mathbb{R}^{d_w} \right\}$

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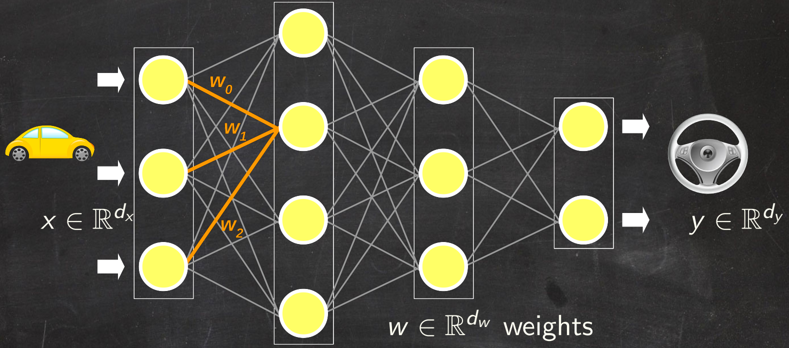


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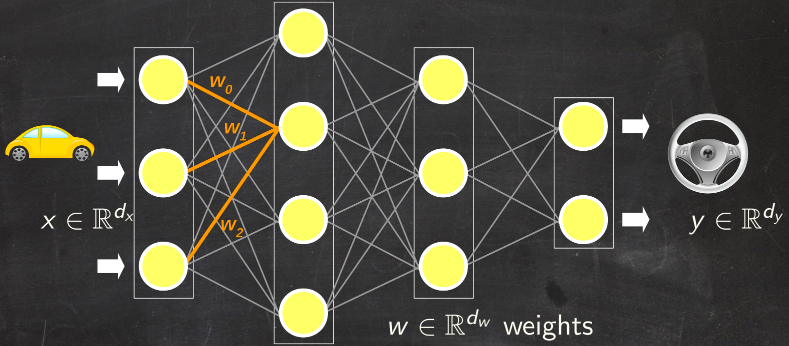


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**Observation**
1. $\Phi$ piecewise smooth $\Rightarrow \mathcal{M}_\Phi$ manifold with singularities
2. $\dim \mathcal{M}_\Phi \leq d_w$

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A **loss function** on a neural network $\Phi : \mathbb{R}^{d_w} \times \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_y}$ is of the form

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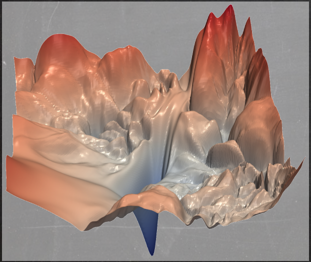
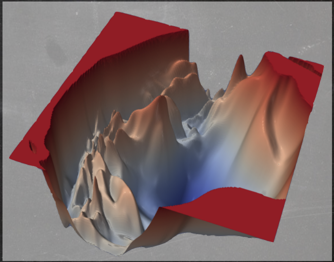


Visualizations of $L$



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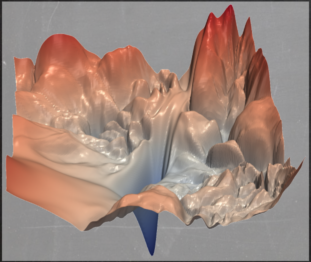

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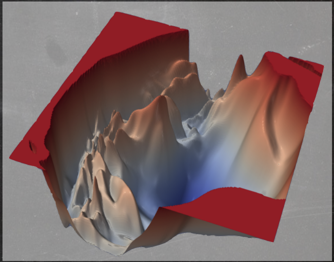

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**Observation** If $\varphi \in \mathrm{Crit}(\ell|_{\mathcal{M}_\Phi})$, then $\mu^{-1}(\varphi) \subset \mathrm{Crit}(L)$.

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Recall: $\mathcal{M}_\Phi = \left\{ M \in \mathbb{R}^{d_h \times d_0} \mid \mathrm{rk}(M) \leq r \right\}$, where $r := \min\{d_0, d_1, \ldots, d_h\}$.

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**Theorem** Let $M \in \mathcal{M}_\Phi$.

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**Corollary** [Baldi & Hornik '89, Kawaguchi '16]
If $\ell$ is a quadratic loss, then all local minima for $L$ are global.
(even in the non-filling case!)

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Fixed data matrices $X \in \mathbb{R}^{d_0 \times s}$ and $Y \in \mathbb{R}^{d_h \times s}$ define a **quadratic loss**

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Minimizing $\ell_{X,Y}$ on the determinantal variety $\mathcal{M}_\Phi = \{M \mid \mathrm{rk}(M) \leq r\}$ is equivalent to minimizing the Euclidean distance of $YX^T$ to $\mathcal{M}_\Phi$.

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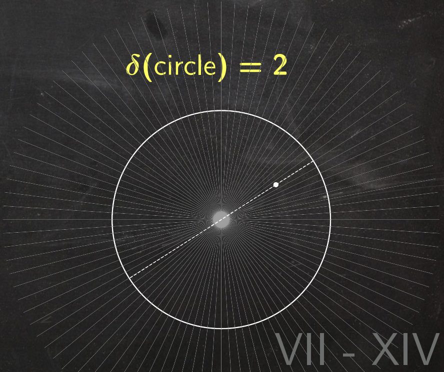

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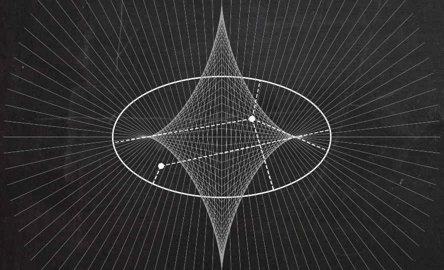

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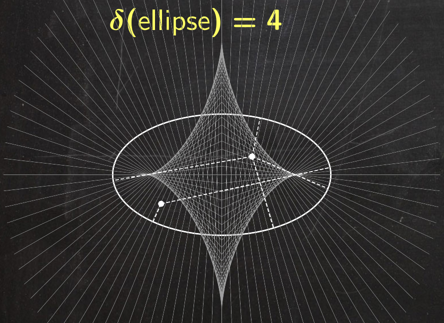


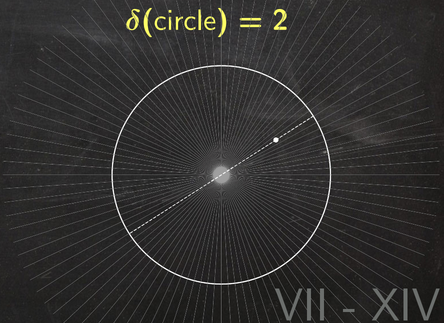

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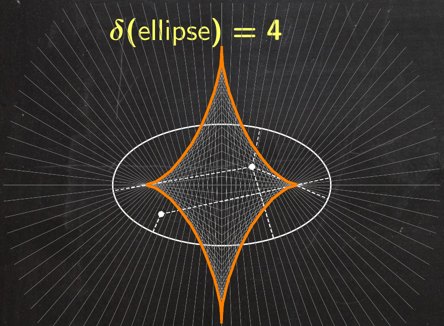


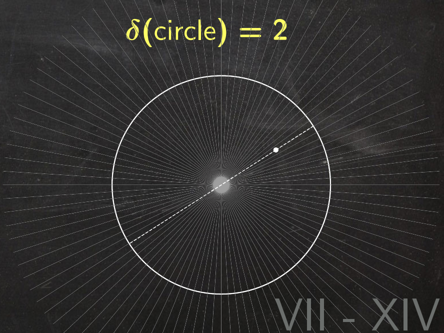

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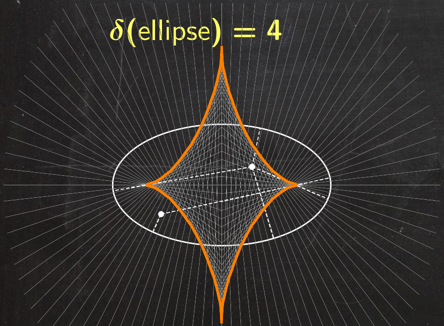


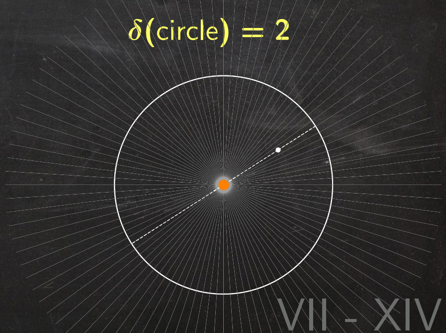

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**Corollary** [Baldi & Hornik '89, Kawaguchi '16]

If $\ell$ is a quadratic loss, then all local minima for the loss $L = \ell \circ \mu$ on a linear network are global. (even in the non-filling case!)

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Let $\mathcal{Z} \subset \mathbb{R}^N$ be an algebraic variety.

There is a constant $\delta^{\mathrm{gen}} \in \mathbb{Z}_{>0}$ such that for almost all linear coordinate changes $f : \mathbb{R}^N \to \mathbb{R}^N$ the ED degree of $f(\mathcal{Z})$ is $\delta^{\mathrm{gen}}$.

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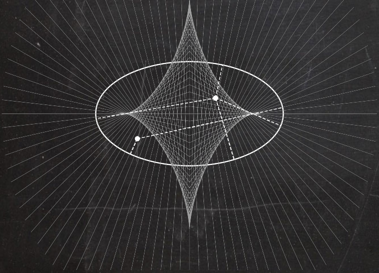

$\delta^{\mathrm{gen}}(\text{circle}) = \delta(\text{ellipse}) = 4$

$\boldsymbol{\delta}(\textbf{circle}) = \mathbf{2}$

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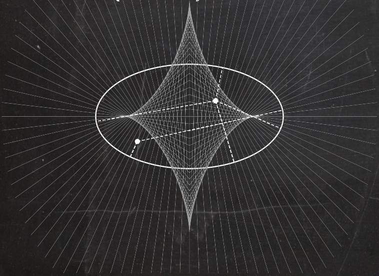

$\delta^{\text{gen}}$(circle)
$= \delta$(ellipse)
$= 4$

$\boldsymbol{\delta}$**(circle) = 2**

Equivalently: $\delta^{\text{gen}}$ is the ED degree of $\mathcal{Z}$
under the perturbed Euclidean distance $\|f(\cdot)\|_2$.

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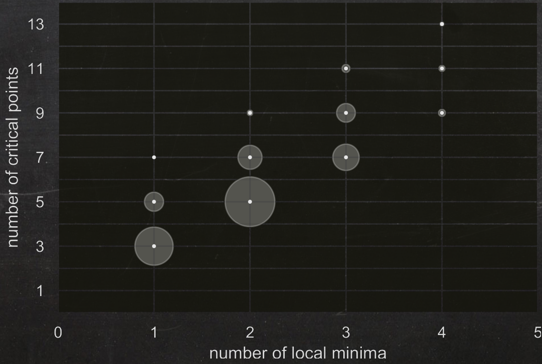
3. Also: different number of local minima in different open regions of $\mathbb{R}^{3\times 3}$, not all of them global !



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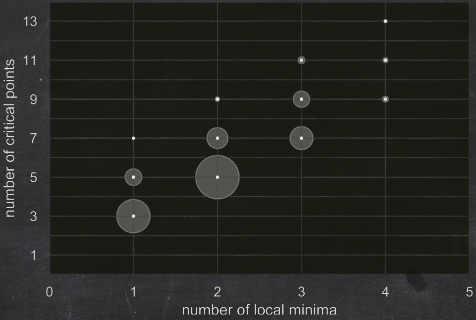


| | | # real critical points | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | **1** | **3** | **5** | **7** | **9** | **11** | **13** |
| # local minima | **1** | 0 | 476 | 120 | 1 | 0 | 0 | 0 |
| | **2** | 0 | 0 | 805 | 190 | 10 | 0 | 0 |
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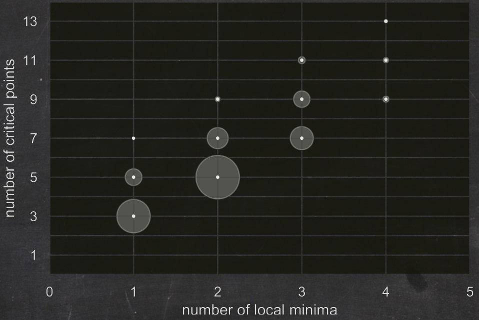


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**All determinantal varieties behave like this !**

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**Remark** Closed formula for generic ED degree of $\mathcal{M}_r = \{M \mid \mathrm{rk}(M) \leq r\} \subset \mathbb{R}^{m \times n}$ involving only $m, n, r$ difficult to derive.

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$$\delta(\mathcal{M}_1) = \min\{m, n\}$$

# Take Away

- determinantal varieties are examples of neuromanifolds
- for linear networks with smooth convex losses:

| | quadratic loss | other loss |
|---|---|---|
| filling | no bad min. | no bad min. |
| non-filling | no bad min. | **bad min.** |

convex optimization on vector space

special embedding of determinantal varieties

- future extensions to
  - convolutional networks (ongoing work with T. Merkh, G. Montúfar, M. Trager)
  - networks with polynomial activation functions or
  - ReLU networks (using semi-algebraic sets)

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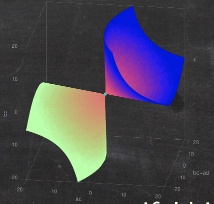


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- In the non-filling case, the neuromanifold is a semi-algebraic set whose boundary is contained in the discriminant hypersurface of polynomials.
- **Example:** If there are 2 filters of even width, the complement of the neuromanifold is a union of two convex cones.