Voronoi Cells of Lattices with Respect to Arbitrary Norms

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joint work with Johannes Blömer (Paderborn University)

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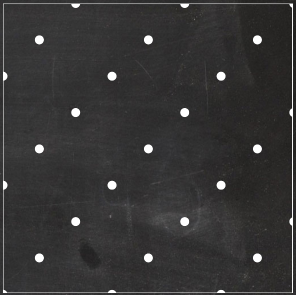
Section 1

Motivation

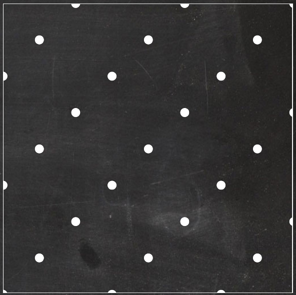
Lattices – 2 equivalent definitions



Definition (I)

An *n-dimensional lattice* is a discrete, additive subgroup of $\mathbb{R}^n$.

Lattices – 2 equivalent definitions



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An *n-dimensional lattice* is a discrete, additive subgroup of $\mathbb{R}^n$.

Definition (II)

Let $b_1, \ldots, b_m \in \mathbb{R}^n$ be linearly independent. Then

$$\mathcal{L}(b_1, \ldots, b_m) := \left\{ \sum_{i=1}^{m} z_i b_i \;\middle|\; z_1, \ldots, z_m \in \mathbb{Z} \right\}$$

is a lattice with basis $(b_1, \ldots, b_m)$ of rank m and dimension n.

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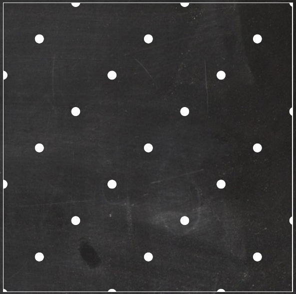

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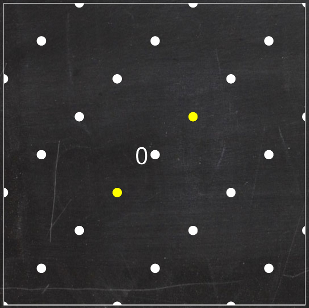
Assume $m = n$.

Lattice problems

Shortest Vector Problem (SVP):
Given lattice basis $(b_1, \ldots, b_n)$, find shortest vector in $\mathcal{L}(b_1, \ldots, b_n) \backslash \{0\}$.



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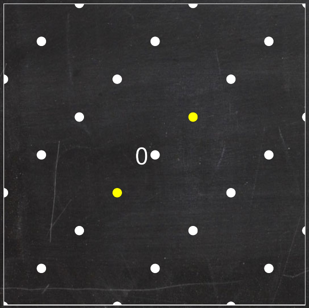
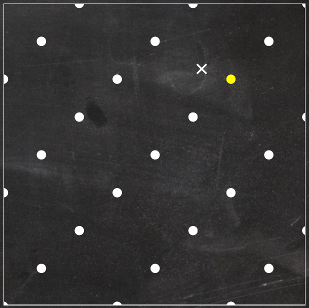


Closest Vector Problem (CVP):
Given lattice basis $(b_1, \ldots, b_n)$ and $x \in \mathbb{R}^n$, find closest vector to x in $\mathcal{L}(b_1, \ldots, b_n)$.



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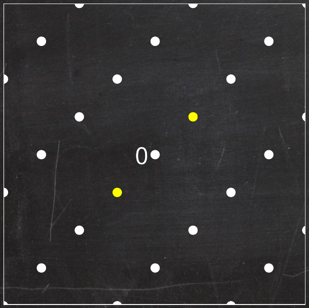

Decision variant NP-hard (under randomized reductions) (Ajtai)

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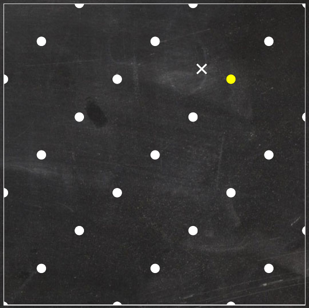
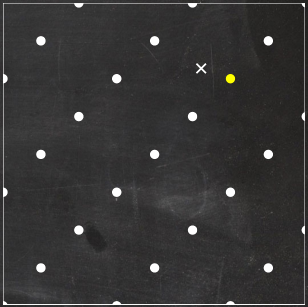


Decision variant NP-complete (Micciancio, Goldwasser)

Lattice problems

Algorithm by Micciancio and Voulgaris:

- solves both problems for Euclidean distance
- $2^{O(n)}$ time and space complexity
- core of algorithm:
  - solve CVP with additional input: Voronoi cell



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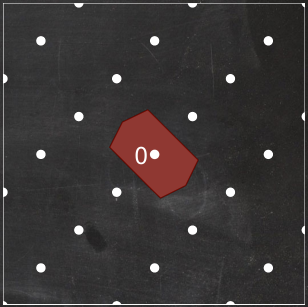

Definition

The Voronoi cell of a lattice Λ w.r.t. a norm $\|\cdot\|$ is

$$\mathcal{V}(\Lambda, \|\cdot\|) := \left\{x \in \mathbb{R}^n \mid \forall v \in \Lambda : \|x\| \leq \|x - v\|\right\}.$$

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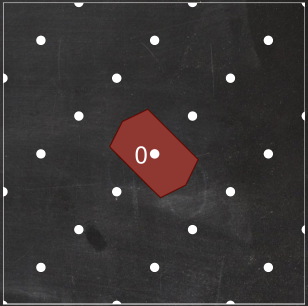

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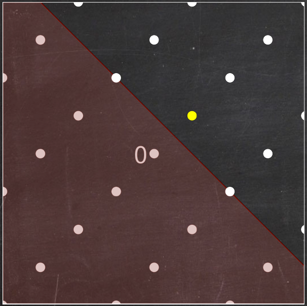
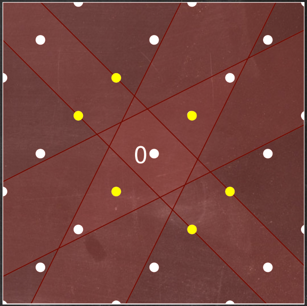
All $v \in \Lambda$ needed?

Voronoi-relevant vectors





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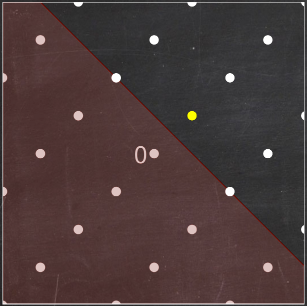

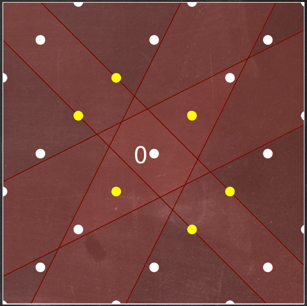

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$v \in \Lambda \backslash \{0\}$ is Voronoi-relevant (VR) w.r.t. $\|\cdot\|$ if $\exists x \in \mathbb{R}^n : \|x\| = \|x - v\|,$
$\forall w \in \Lambda \backslash \{0, v\} : \|x\| < \|x - w\|.$

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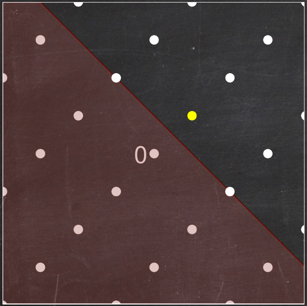

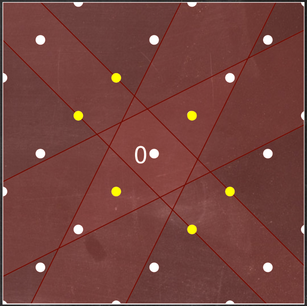

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$\Rightarrow$ VR vectors determine Voronoi cell for Euclidean norm $\| \cdot \|_2$, i.e.,
$\mathcal{V}(\Lambda, \| \cdot \|_2) = \{x \in \mathbb{R}^n \mid \forall v \in \Lambda \ : \|x\|_2 \leq \|x - v\|_2\}$ (Agrell et al.)

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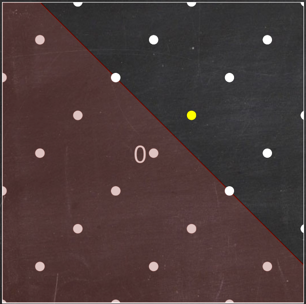

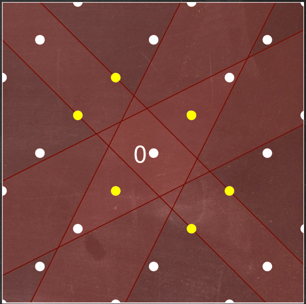

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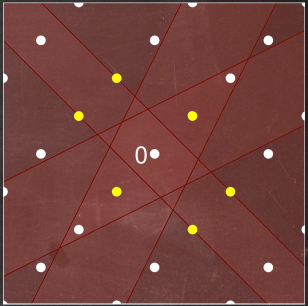

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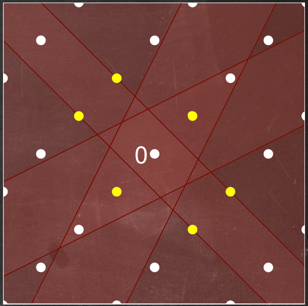

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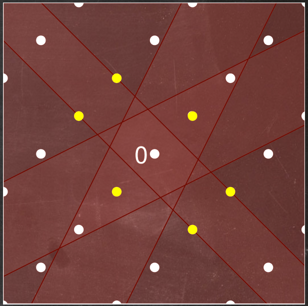

- at most $2(2^n - 1)$ Voronoi-relevant vectors in n-dimensional lattice w.r.t. Euclidean norm (Agrell et al.)
  - essential for above algorithm

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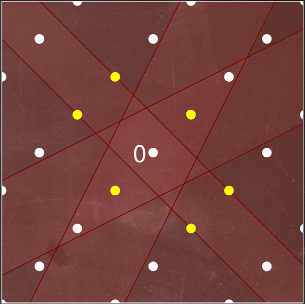

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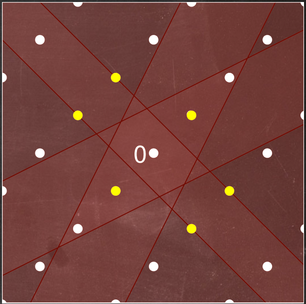

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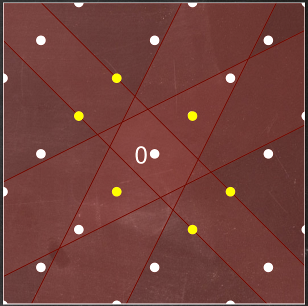

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$\Longrightarrow$ **Upper bound for number of Voronoi-relevant vectors w.r.t. arbitrary p-norms?**

Section 2

2 Dimensions

Geometry

Do VR vectors determine Voronoi cell?

Combinatorics

How many $v \in \Lambda$ are VR w.r.t. $\|\cdot\|$?

Geometry

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$n \geq 1$: yes, for Euclidean norm

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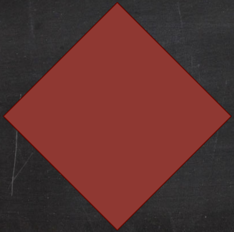
Our results:

$n = 2$: yes, for strictly convex norm

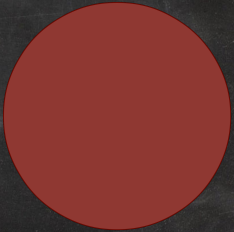
4 or 6

Strict convexity

Definition
A norm is strictly convex if its unit sphere does not contain a line segment.



$\|\cdot\|_1$: not strictly convex



$\|\cdot\|_2$: strictly convex

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no, for non-strictly convex norm
$\Rightarrow$ Generalized Voronoi-relevant (GVR) vectors determine Voronoi cell

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#GVR vectors: not const.

Section 3

General Dimensions

Geometry

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#GVR vectors: not bounded by $f(n)$

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$n \geq 2$: GVR vectors determine Voronoi cell

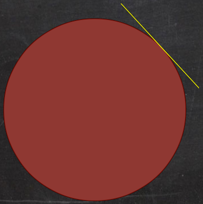
#GVR vectors: not bounded by $f(n)$

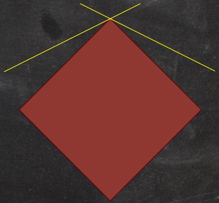
yes, for strictly convex and smooth norm
$\Rightarrow$ bijection between VR vectors and facets of Voronoi cell

Smooth norms

Definition
Let $S \subseteq \mathbb{R}^n$ and $s \in \partial S$. A hyperplane $H \in \mathbb{R}^n$ is a supporting hyperplane of S at s if

- $s \in H$ and
- S is contained in one of the 2 closed halfspaces bounded by H





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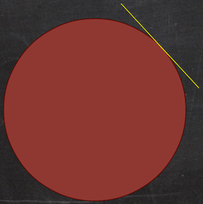
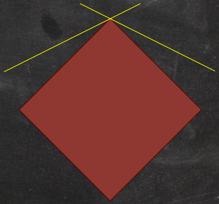
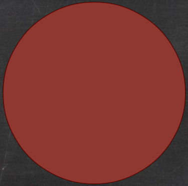




Definition
A norm is smooth if each point on its unit sphere has a unique supporting hyperplane.

Smooth norms

	smooth	not smooth
strictly convex		
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There is no upper bound for the number of Voronoi-relevant vectors

- **w.r.t. general strictly convex and smooth norms**
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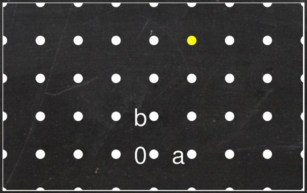
Q: Can $a + mb$ for $a, b \in \Lambda$ and large $m \in \mathbb{N}$ be Voronoi-relevant?



$m = 3$

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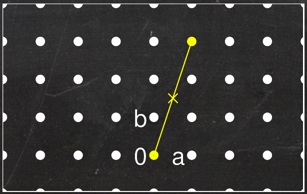
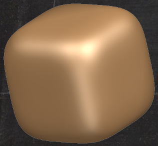


$m = 3$

3-norm



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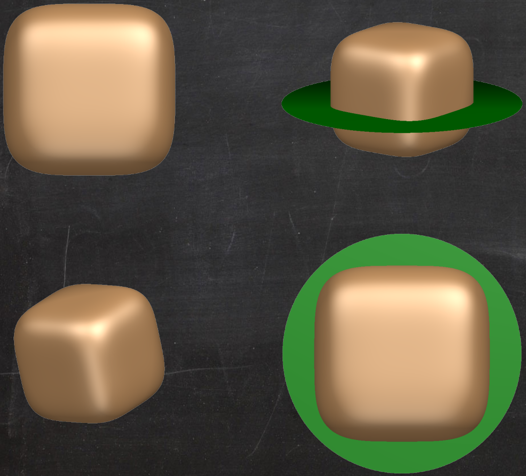

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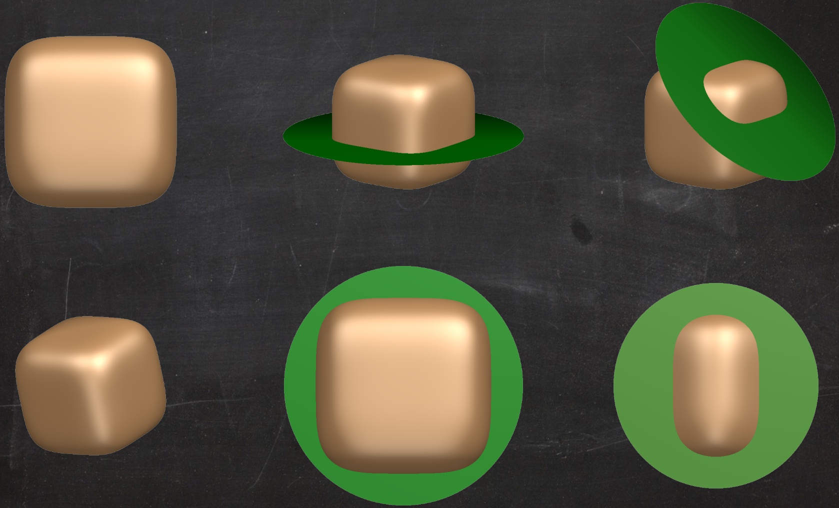
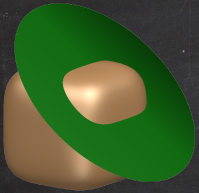
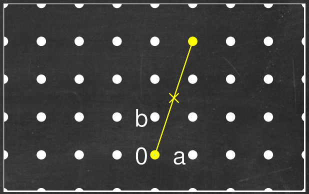
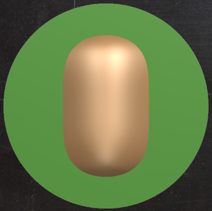


Idea







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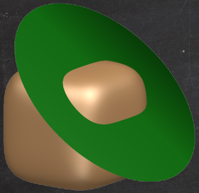
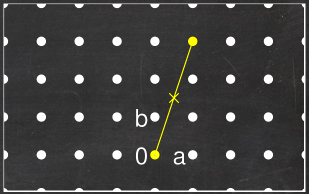
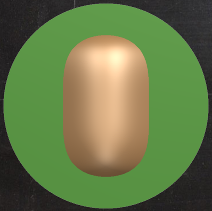
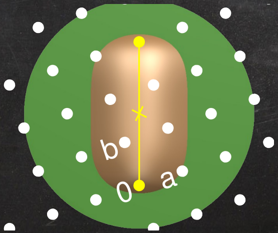


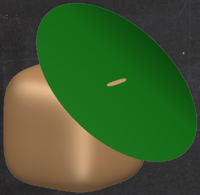


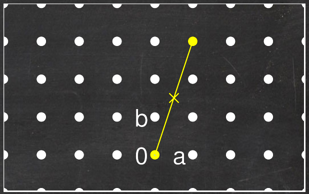
Rotate lattice s.t.





Idea





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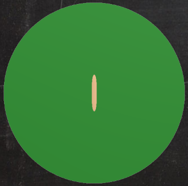
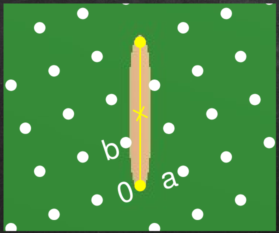
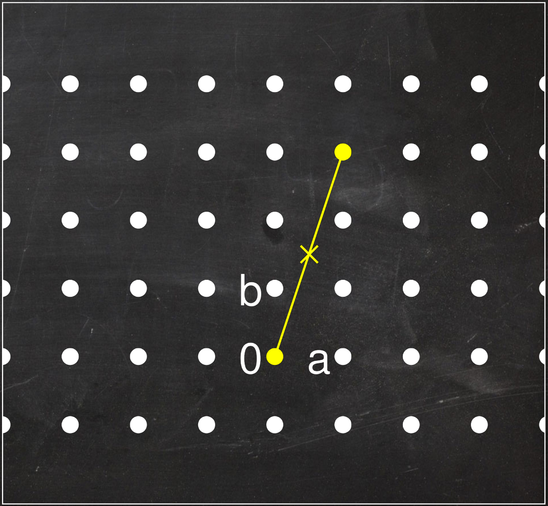




Construct lattice family Λ_m

Modify standard lattice:



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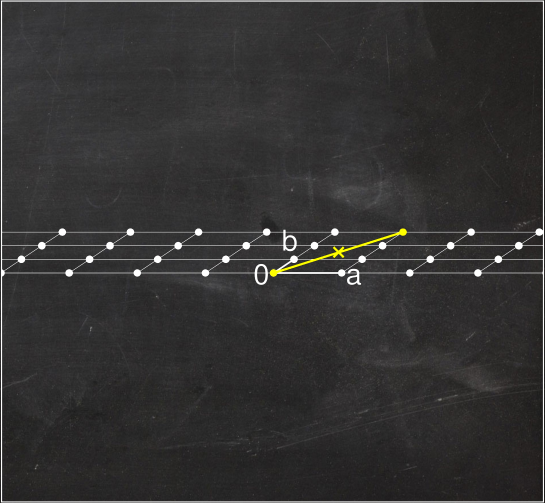

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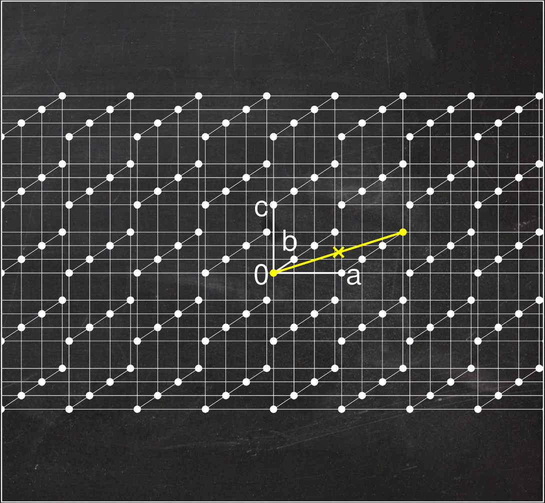
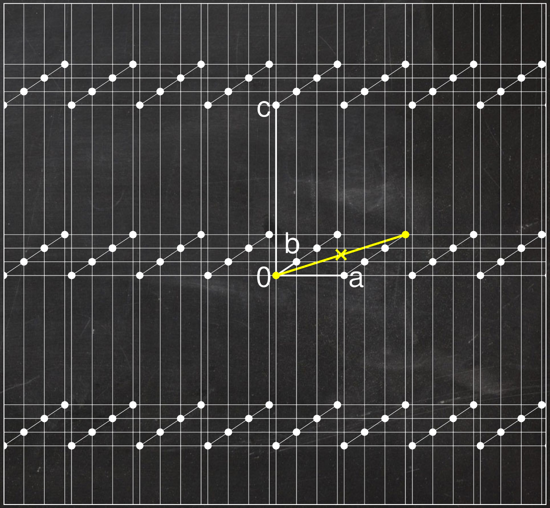


Construct lattice family Λ_m

Modify standard lattice:

- Stretch in c-direction by $5\sqrt{2}m^5$

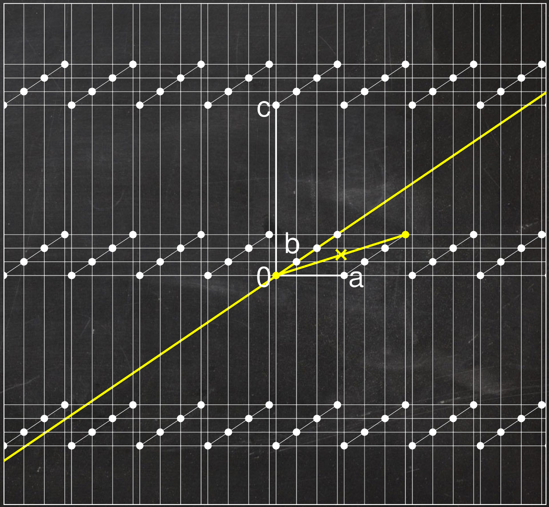


Construct lattice family Λ_m

Modify standard lattice:

I Stretch in c-direction by $5\sqrt{2}m^5$

II Rotate around c-axis s.t. $a + mb$ lies on yellow axis



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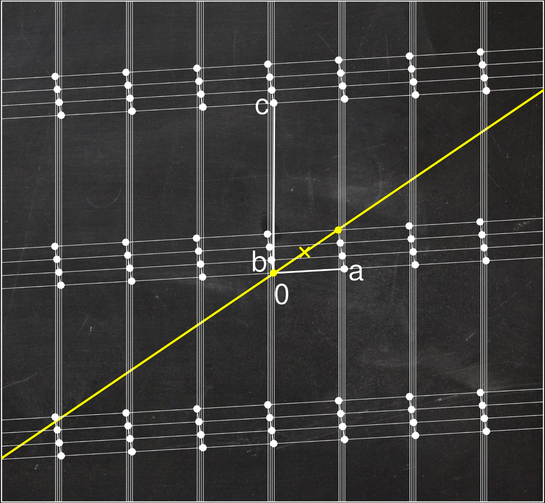

Construct lattice family Λ_m

Modify standard lattice:

I Stretch in c-direction by $5\sqrt{2}m^5$

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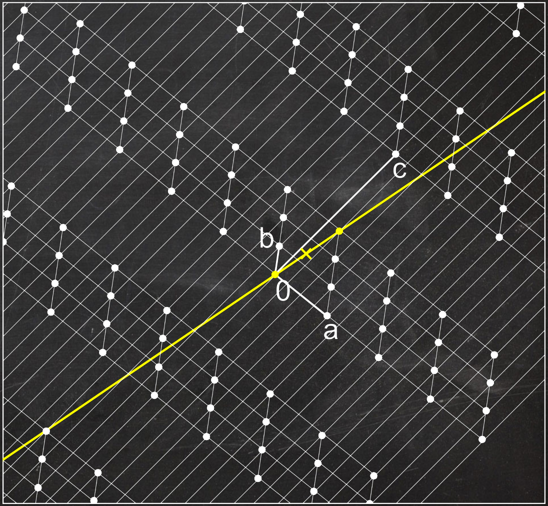
III Rotate around yellow axis by 45°



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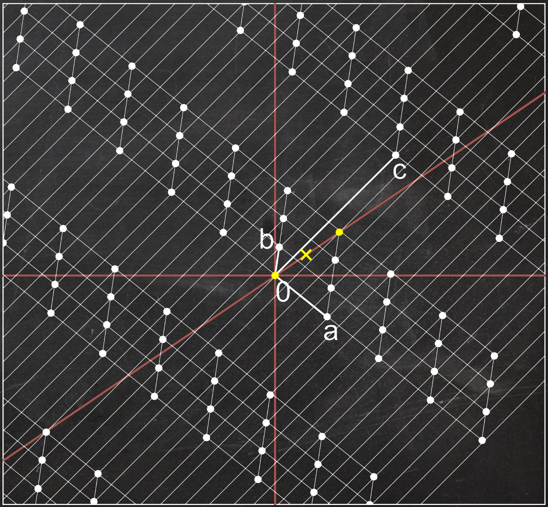
$\Longrightarrow$ Lattice Λ_m



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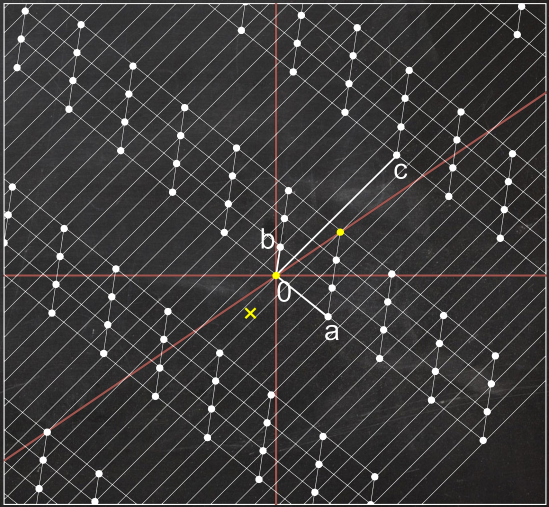
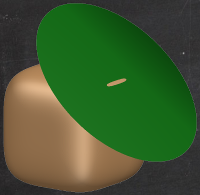
$\Longrightarrow$ Lattice Λ_m

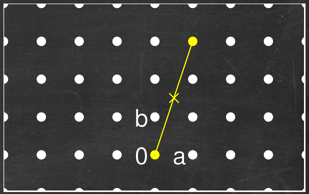
- Move x along c-direction

$\Longrightarrow$ $a + mb$ VR w.r.t. 3-norm



Construct lattice family Λ_m





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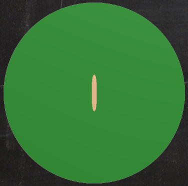

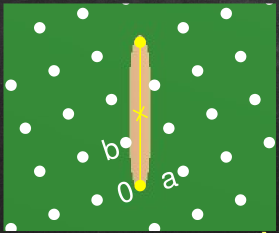

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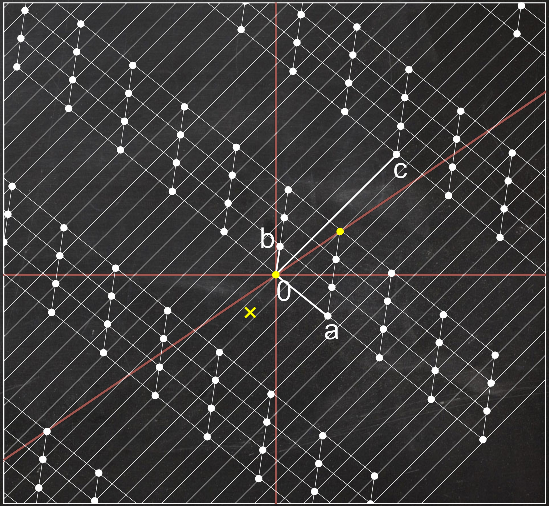
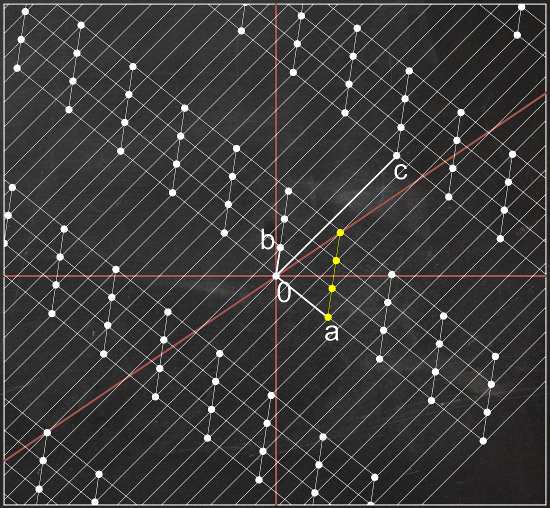


Construct lattice family Λ_m

Analogous:

Theorem (Blömer, K.)
For $2 \leq k \leq \sqrt{m}$, $a + kb$ is Voronoi-relevant in Λ_m w.r.t. 3-norm.



Geometry

Do VR vectors determine Voronoi cell?

$n \geq 1$: yes, for Euclidean norm

Combinatorics

How many $v \in \Lambda$ are VR w.r.t. $\|\cdot\|$?

$\leq 2(2^n - 1)$

Our results:

$n = 2$: yes, for strictly convex norm
$\Rightarrow$ bijection between VR vectors and facets of Voronoi cell

no, for non-strictly convex norm

4 or 6

$n \geq 2$: GVR vectors determine Voronoi cell

yes, for strictly convex and smooth norm
$\Rightarrow$ bijection between VR vectors and facets of Voronoi cell

#GVR vectors:
not bounded by $f(n)$

not bounded by $f(n)$

Thank you!

Geometry

Do VR vectors determine Voronoi cell?

$n \geq 1$: yes, for Euclidean norm

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Combinatorics

How many $v \in \Lambda$ are VR w.r.t. $\|\cdot\|$?

$\leq 2(2^n - 1)$

4 or 6

#GVR vectors: not bounded by $f(n)$

not bounded by $f(n)$

but by $\left(1 + 4\frac{\mu(\Lambda,\|\cdot\|)}{\lambda_1(\Lambda,\|\cdot\|)}\right)^n$

General upper bound

Proposition (Blömer, K.)

Every lattice $\Lambda \subseteq \mathbb{R}^n$ has at most $\left(1 + 4\frac{\mu(\Lambda,\|\cdot\|)}{\lambda_1(\Lambda,\|\cdot\|)}\right)^n$ generalized Voronoi-relevant vectors w.r.t. every norm.

Definition

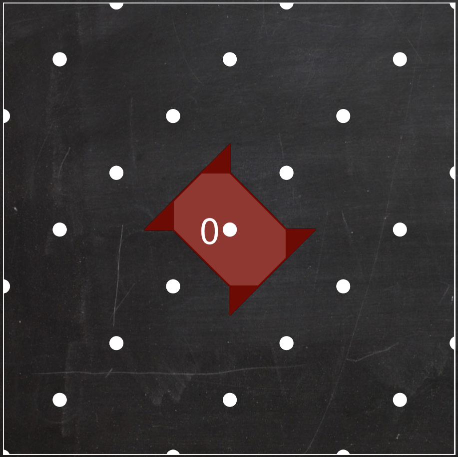
The covering radius of Λ w.r.t. $\|\cdot\|$ is

$$\mu(\Lambda, \|\cdot\|) := \inf\{d \in \mathbb{R}_{\geq 0} \mid \forall x \in \mathbb{R}^n \, \exists v \in \Lambda : \|x - v\| \leq d\}.$$

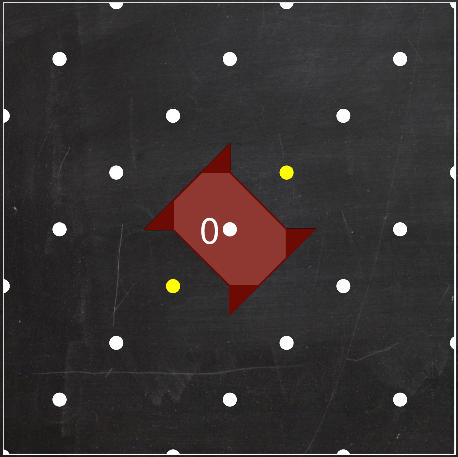
The first successive minimum of Λ w.r.t. $\|\cdot\|$ is

$$\lambda_1(\Lambda, \|\cdot\|) := \inf\{\|v\| \mid v \in \Lambda, v \neq 0\}.$$

Taxicab norm $\|\cdot\|_1$ – Voronoi cell

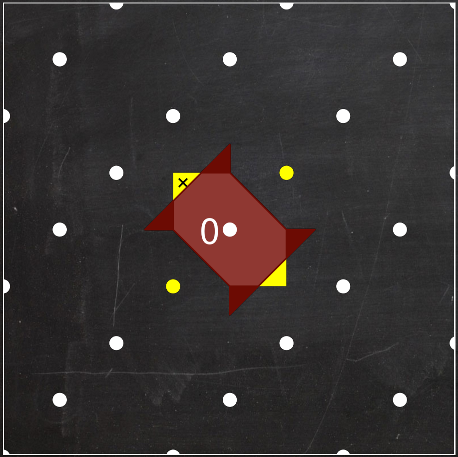


Taxicab norm $\|\cdot\|_1$ – Voronoi cell



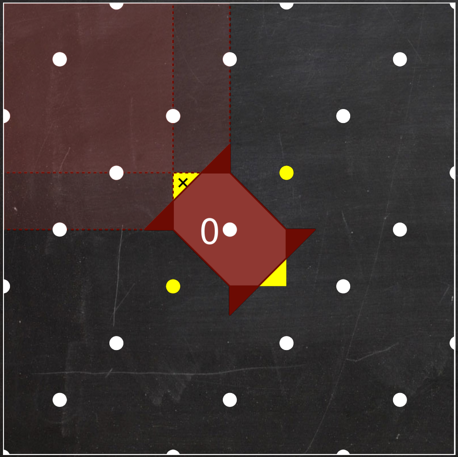
- 2 Voronoi-relevant vectors

Taxicab norm $\|\cdot\|_1$ – Voronoi cell



- 2 Voronoi-relevant vectors
- x not in Voronoi-cell

Taxicab norm $\|\cdot\|_1$ – Voronoi cell



- 2 Voronoi-relevant vectors
- x not in Voronoi-cell, BUT:
- x closer to 0 than to Voronoi-relevant vectors

Generalized Voronoi-relevant vectors

Definition
$v \in \Lambda \backslash \{0\}$ is Voronoi-relevant (VR) w.r.t. $\| \cdot \|$ if

$$\begin{aligned} \exists x \in \mathbb{R}^n &: \|x\| = \|x - v\|, \\ \forall w \in \Lambda \backslash \{0, v\} &: \|x\| < \|x - w\|. \end{aligned}$$

Generalized Voronoi-relevant vectors

Definition

$v \in \Lambda \setminus \{0\}$ is generalized Voronoi-relevant (GVR) w.r.t. $\|\cdot\|$ if

$$\exists x \in \mathbb{R}^n : \|x\| = \|x - v\|,$$
$$\forall w \in \Lambda : \|x\| \leq \|x - w\|.$$

Generalized Voronoi-relevant vectors

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$v \in \Lambda \backslash \{0\}$ is generalized Voronoi-relevant (GVR) w.r.t. $\|\cdot\|$ if

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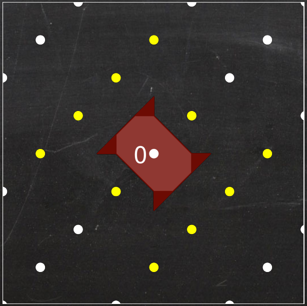
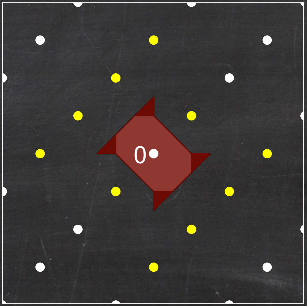


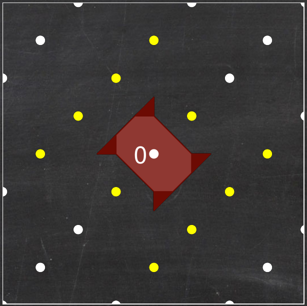
Generalized Voronoi-relevant vectors



Theorem (Blömer, K.)

- *For every lattice* $\Lambda \subseteq \mathbb{R}^n$ *and every norm* $\|\cdot\|$*,*
  $\mathcal{V}(\Lambda, \|\cdot\|) = \{x \in \mathbb{R}^n \mid \forall v \in \Lambda \textbf{ GVR} : \|x\| \leq \|x - v\|\}.$

Generalized Voronoi-relevant vectors



$m = 3$

Theorem (Blömer, K.)

I *For every lattice* $\Lambda \subseteq \mathbb{R}^n$ *and every norm* $\|\cdot\|$,
$\mathcal{V}(\Lambda, \|\cdot\|) = \{x \in \mathbb{R}^n \mid \forall v \in \Lambda$ ***GVR*** $: \|x\| \leq \|x - v\|\}$.

II $\mathcal{L}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\m\end{pmatrix}\right)$ *has at least* $2m$ *GVR vectors w.r.t. Taxicab norm.*