The geometry of neural networks

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joint with Matthew Trager and Joan Bruna
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Neural Networks

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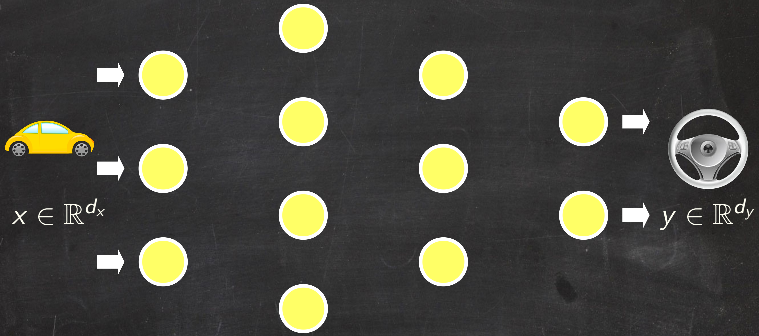
$x \in \mathbb{R}^{d_x}$

$y \in \mathbb{R}^{d_y}$

Neural Networks



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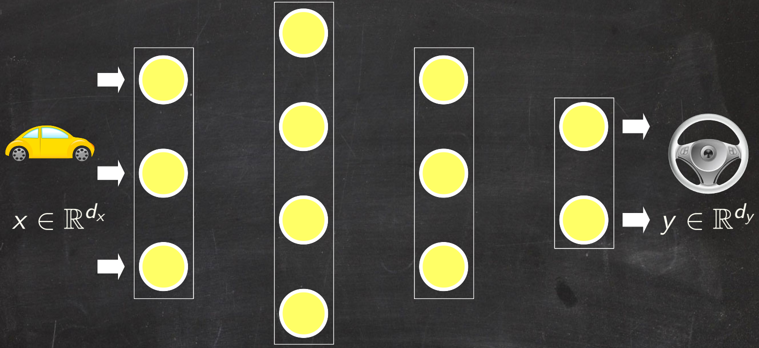

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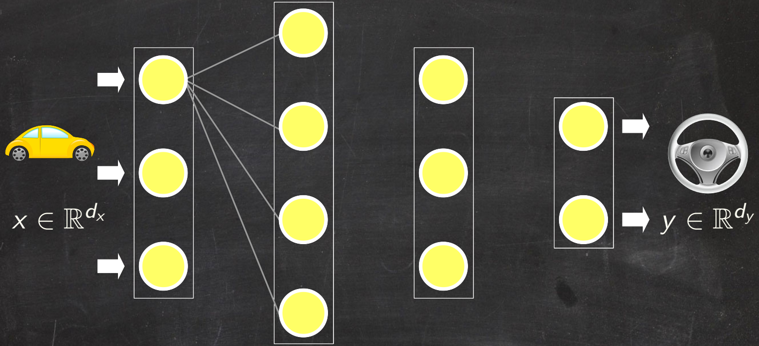

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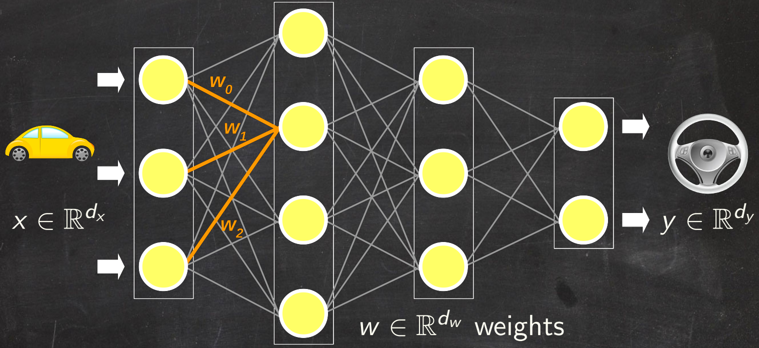

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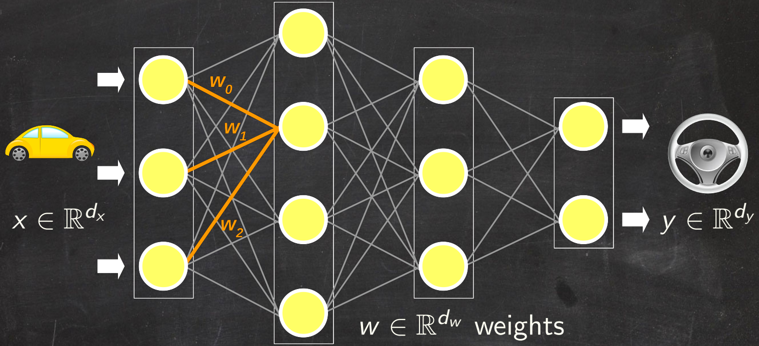

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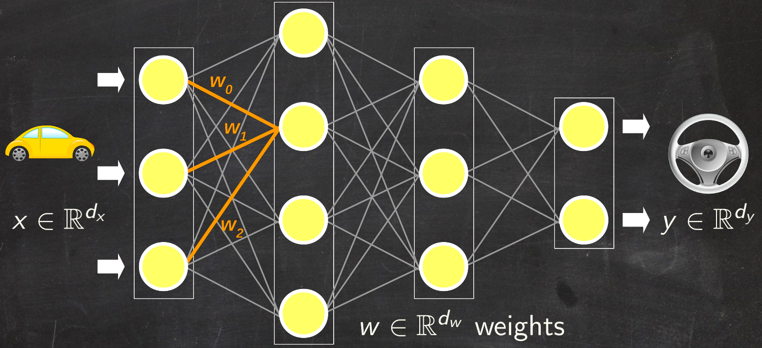

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Definition $\mathcal{M}_\Phi := \left\{ \Phi(w, \cdot) : \mathbb{R}^{d_x} \to \mathbb{R}^{d_y} \mid w \in \mathbb{R}^{d_w} \right\}$

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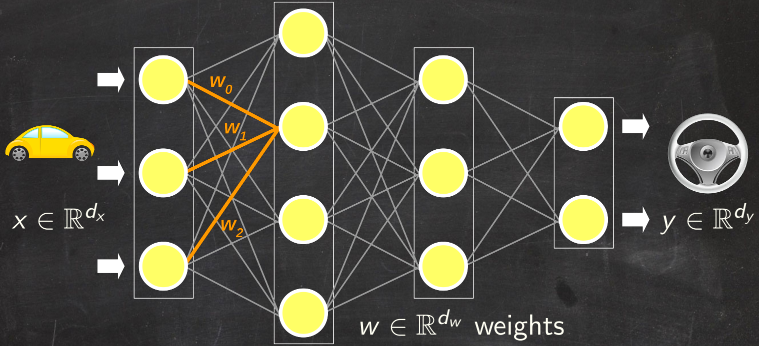

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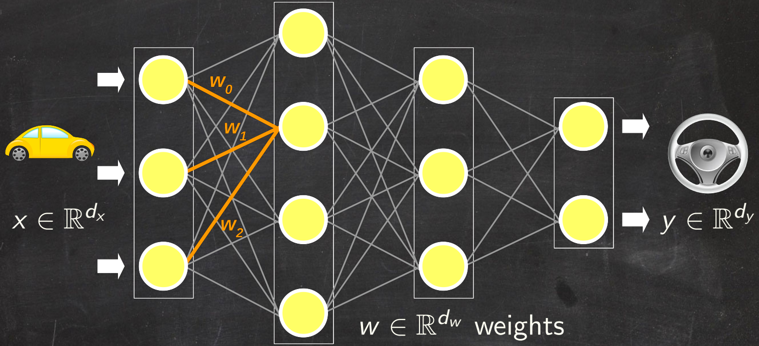

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Observation

1. Φ piecewise smooth $\Rightarrow \mathcal{M}_\Phi$ manifold with singularities
2. $\dim \mathcal{M}_\Phi \leq d_w$

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A **linear network** is defined by a map $\Phi : \mathbb{R}^{d_w} \times \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_y}$ of the form

$$\Phi(w, x) = W_h W_{h-1} \dots W_1 x,$$

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Example The neuromanifold of the linear network Φ is

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Loss Landscapes

A **loss function** on a neural network $\Phi : \mathbb{R}^{d_w} \times \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_y}$ is of the form

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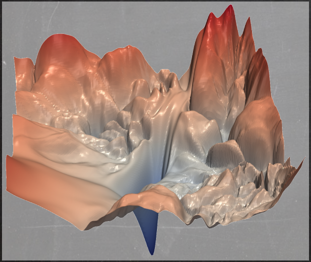
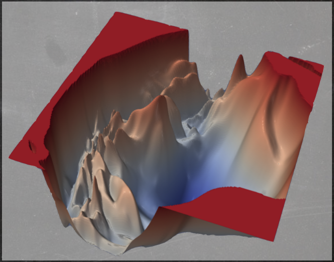


Visualizations of L



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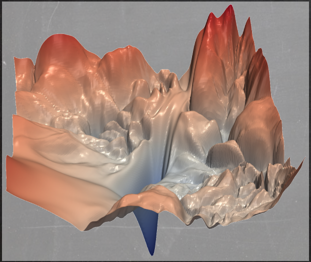

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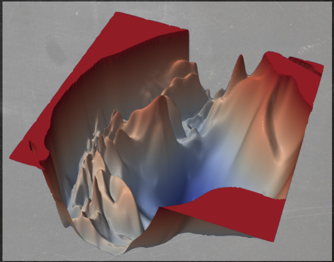

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Observation If $\varphi \in \mathrm{Crit}(\ell|_{\mathcal{M}_\Phi})$, then $\mu^{-1}(\varphi) \subset \mathrm{Crit}(L)$.

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Theorem Let $M \in \mathcal{M}_\Phi$.

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$$\text{where } b := \#\{i \mid 0 < i < h, d_i = r\}.$$

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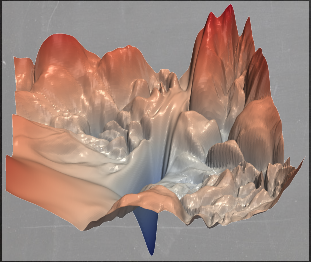

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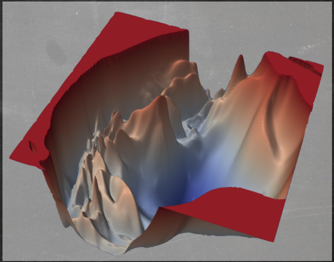

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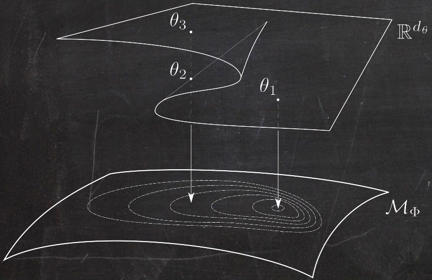

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$w^* \in \mathrm{Crit}(L)$ is called **pure** if $\mu(w^*) \in \mathrm{Crit}(\ell|_{\mathcal{M}_\Phi})$ and $\mu(w^*) \notin \mathrm{Sing}\,\mathcal{M}_\Phi$.

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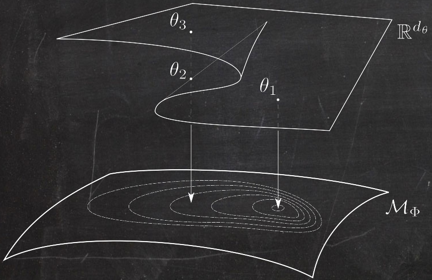

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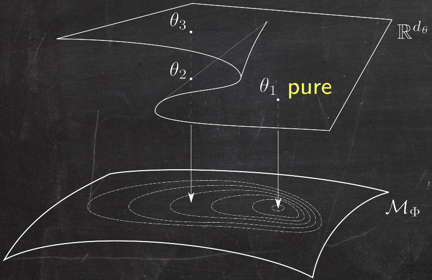

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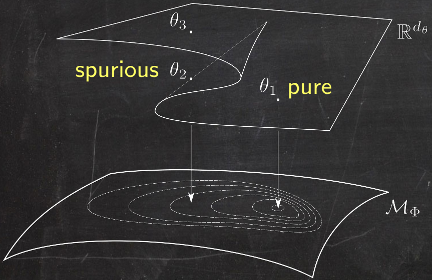

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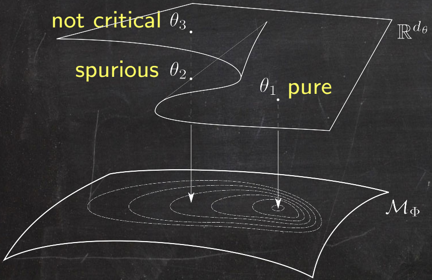

Definition
$w^* \in \mathrm{Crit}(L)$ is called **pure** if $\mu(w^*) \in \mathrm{Crit}(\ell|_{\mathcal{M}_\Phi})$ and $\mu(w^*) \notin \mathrm{Sing}\,\mathcal{M}_\Phi$.
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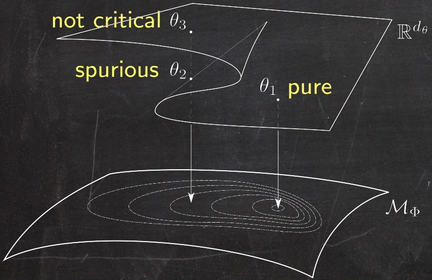

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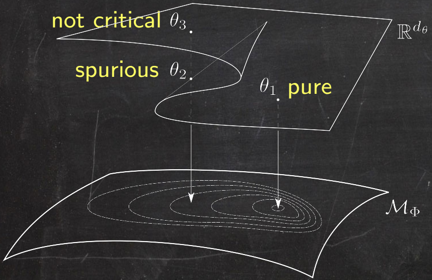

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Linear Networks

A loss function on a linear network is of the form

$$L : \mathbb{R}^{d_h \times d_{h-1}} \times \ldots \times \mathbb{R}^{d_1 \times d_0} \xrightarrow{\quad\mu\quad} \mathcal{M}_\Phi \subset \mathbb{R}^{d_h \times d_0} \xrightarrow{\quad\ell\quad} \mathbb{R},$$

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(even in the non-filling case!)

The Quadratic Loss

Fixed data matrices $X \in \mathbb{R}^{d_0 \times s}$ and $Y \in \mathbb{R}^{d_h \times s}$ define a **quadratic loss**

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Minimizing $\ell_{X,Y}$ on the determinantal variety $\mathcal{M}_\Phi = \{M \mid \mathrm{rk}(M) \leq r\}$ is equivalent to minimizing the Euclidean distance of YX^T to $\mathcal{M}_\Phi$.

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Let $\mathcal{Z} \subset \mathbb{C}^N$ be an algebraic variety
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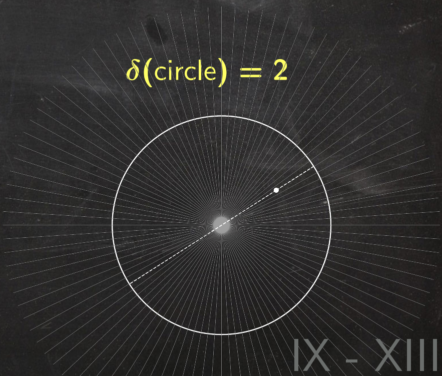

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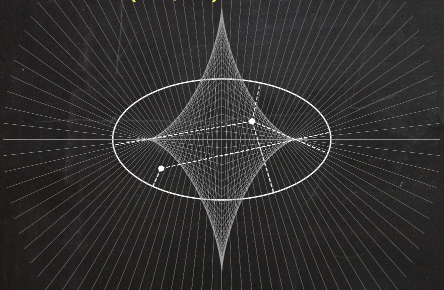

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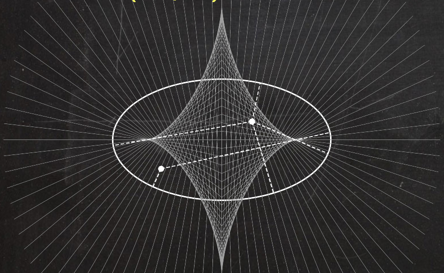

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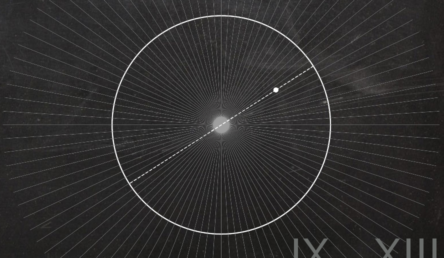

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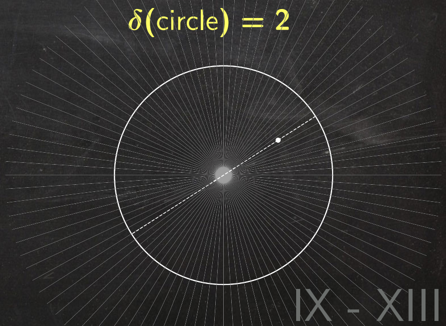

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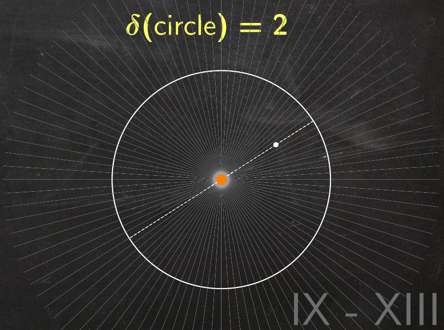

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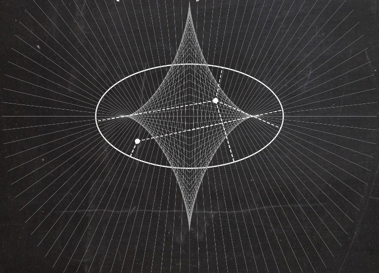

$$\delta^{\mathrm{gen}}(\text{circle}) = \delta(\text{ellipse}) = 4$$

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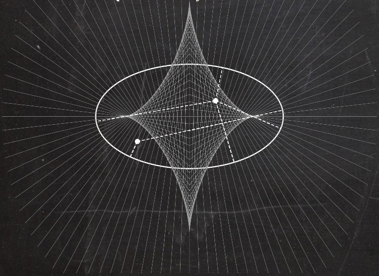

δ^{gen}(circle)
$= \delta$(ellipse)
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Equivalently: δ^{gen} is the ED degree of $\mathcal{Z}$
under the perturbed Euclidean distance $\|f(\cdot)\|_2$.

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Example $\mathcal{M}_1 = \{M \mid \mathrm{rk}(M) \leq 1\} \subset \mathbb{C}^{3\times 3}$

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$$\delta(\mathcal{M}_1) = \min\{m, n\}$$

Take Away

- neuromanifolds
- pure & spurious critical points vs. bad minima
- for linear networks with smooth convex losses:

	quadratic loss	other loss
filling	no bad min.	no bad min.
non-filling	no bad min.	**bad min.**

convex optimization on vector space

special embedding of determinantal varieties

- future extensions to
  - networks with polynomial activation functions or
  - ReLU networks (using semi-algebraic sets)