A Lower Bound for Computing the Diameter

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Diameter

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Definition

Let $G = (V, E)$ be a graph, $u, v \in V$.

- $n := |V|$, $m := |E|$
- distance: $d(u, v) \hat{=}$ length of shortest path between u and v
- diameter: $\operatorname{diam}(G) := \max_{u,v \in V} d(u, v)$

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- breadth-first search from every $v \in V$
  $\Rightarrow$ running time: $O(n \cdot (n + m))$

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Computing the diameter:

- breadth-first search from every $v \in V$
  $\Rightarrow$ running time: $O(n \cdot (n + m))$
- connected G, algorithm using matrix multiplication
  $\Rightarrow$ running time: $O(\underbrace{M(n)}_{\text{time for } n \times n\text{-matrix multiplication, } M(n) \in O(n^{2.3727})} \log n)$

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Theorem

$\forall n \geq 10\ \forall B \geq 1\ \forall \epsilon > 0$ sufficiently small: $R_\epsilon^{dc}(\mathrm{diam}_4) \in \Omega(\frac{n}{B})$ (even when diameter is bounded by 5)

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Idea of proof:

- part I: transfer to another model of computation
- part II: use known lower bound

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Model of computation: Communication

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- given $a \in \{0,1\}^k$ to Alice, $b \in \{0,1\}^k$ to Bob
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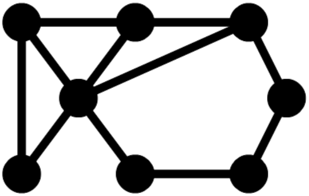
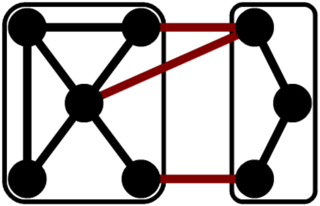
Definition

Let $G = (V, E)$ be a graph. (G_a, G_b, C) with subgraphs $G_a = (V_a, E_a)$ and $G_b = (V_b, E_b)$ is a cut iff $V = V_a \dot\cup V_b$, $E = E_a \dot\cup E_b \dot\cup C$ and $C = \{\{u, v\} \in E \mid u \in V_a, v \in V_b\}$.





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Let G be a graph with cut (G_a, G_b, C). Define

$$f'((G_a, C), (G_b, C)) := f(G).$$

Part I: transfer to another model of computation

Lemma

If $f(G)$ can be computed in the DistributedRound(B) model, $f'((G_a, C), (G_b, C)) := f(G)$ can be computed in the Communication model. Furthermore

$$\frac{R_\epsilon^{cc}(f')}{2|C|B} \leq R_\epsilon^{dc}(f).$$

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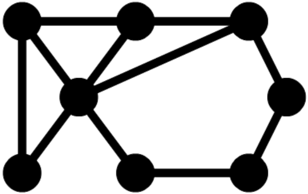

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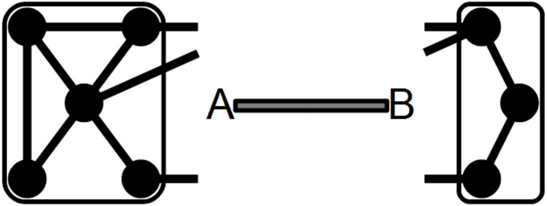

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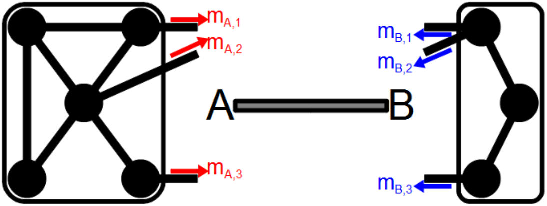

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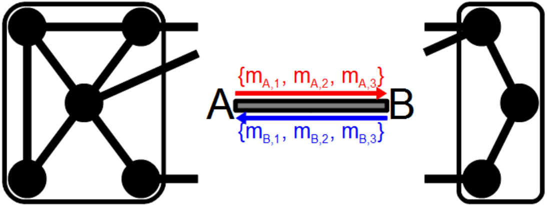

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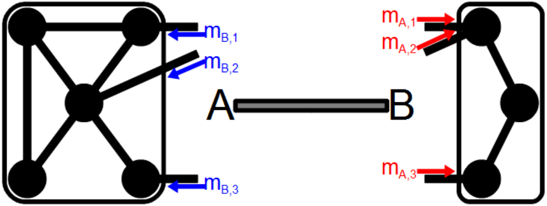

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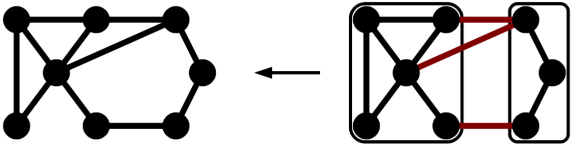

$f(G)$ $\qquad$ $f'((G_a, C), (G_b, C))$

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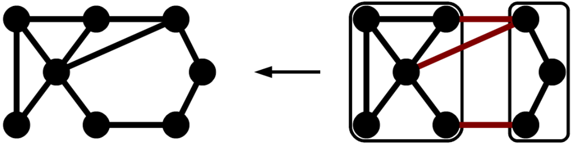
Theorem

$\forall n \geq 10 \; \forall B \geq 1 \; \forall \epsilon > 0$ *sufficiently small:* $R_\epsilon^{dc}(\text{diam}_4) \in \Omega(\frac{n}{B})$ *(even when diameter is bounded by 5)*



$\text{diam}_4(G)$ $\qquad$ $\text{diam}'_4((G_a, C), (G_b, C))$

$$R_\epsilon^{dc}(\text{diam}_4) \geq \frac{R_\epsilon^{cc}(\text{diam}'_4)}{2|C|B}$$

Part II: use known lower bound

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$\text{disj}_k : \{0,1\}^k \times \{0,1\}^k \to \{0,1\}$,

$$\text{disj}_k(a,b) := \begin{cases} 0 & , \exists i \in \{0,\ldots,k-1\} : a_i = b_i = 1 \\ 1 & , \textit{else} \end{cases}$$

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$\forall \epsilon > 0$ *sufficienty small:* $R^{cc}_{\epsilon}(\text{disj}_k) \in \Omega(k)$

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Needed:
$\mathcal{R}_k : \{\mathrm{A},\mathrm{B}\} \times \{0,1\}^k \to \{(H,C) \mid H \subseteq G, (H, G \setminus H, C) \text{ cut}\}$
such that $\mathrm{disj}_k(a,b) = \mathrm{diam}'_4(\mathcal{R}_k(A,a), \mathcal{R}_k(B,b))$

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$$\Rightarrow R^{dc}_\epsilon(\text{diam}_4) \geq \frac{R^{cc}_\epsilon(\text{diam}'_4)}{2|C|B} \geq \frac{R^{cc}_\epsilon(\text{disj}_k)}{2|C|B}$$

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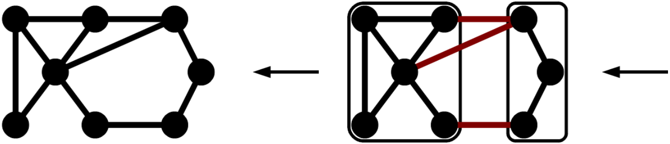

Alice: a
Bob: b

$\mathrm{diam}_4(G)$ | $\mathrm{diam}'_4((G_a, C), (G_b, C))$ | $\mathrm{disj}_k(a, b)$

$$R^{dc}_{\epsilon}(\mathrm{diam}_4) \geq \frac{R^{cc}_{\epsilon}(\mathrm{diam}'_4)}{2|C|B} \geq \frac{R^{cc}_{\epsilon}(\mathrm{disj}_k)}{2|C|B}$$

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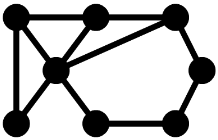

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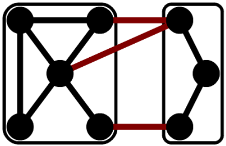

←

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$\text{diam}_4(G)$ $\quad$ $\text{diam}_4'((G_a, C), (G_b, C))$ $\quad$ $\text{disj}_{k_n^2}(a, b)$

$$R_\epsilon^{dc}(\text{diam}_4) \geq \frac{R_\epsilon^{cc}(\text{diam}_4')}{2|C|B} \geq \frac{R_\epsilon^{cc}(\text{disj}_{k_n^2})}{2|C|B}$$

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Needed:
$\mathcal{R}_{k_n^2} : \{\text{A, B}\} \times \{0,1\}^{k_n^2} \rightarrow \{(H, C) \mid H \subseteq G, (H, G \setminus H, C) \text{ cut}\}$
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$k_n := \lfloor \frac{n}{10} \rfloor$	Ex.: $n = 20, k_n = 2,$ $a = (0,0,0,1), b = (0,1,1,1)$

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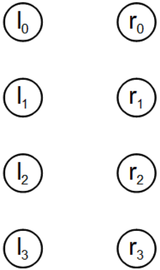

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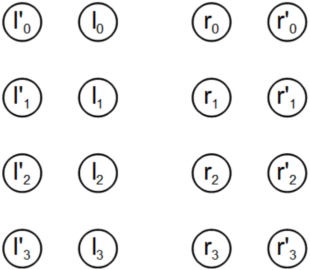

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c_L	$c_R, W := \{w_i \mid i < n - 8k_n - 2\}$

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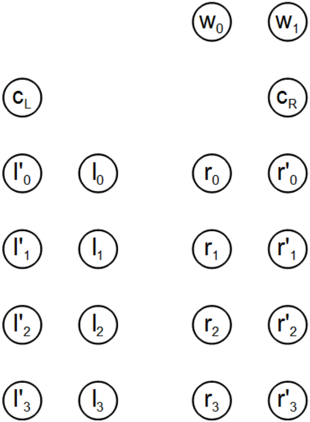
Alice Bob



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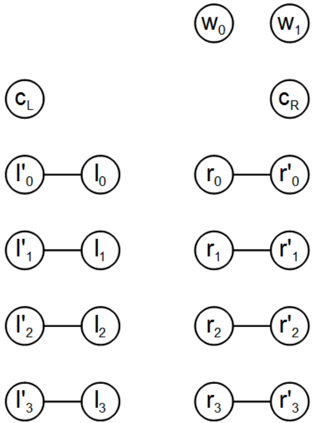
$L := \{l_i \mid i < 2k_n\}$ $L' := \{l'_i \mid i < 2k_n\}$ c_L	$R := \{r_i \mid i < 2k_n\}$ $R' := \{r'_i \mid i < 2k_n\}$ $c_R, W := \{w_i \mid i < n - 8k_n - 2\}$
$E_A := \{\{l_i, l'_i\} \mid i < 2k_n\}$	$E_B := \{\{r_i, r'_i\} \mid i < 2k_n\}$

Ex.: $n = 20, k_n = 2,$
$a = (0, 0, 0, 1), b = (0, 1, 1, 1)$

Alice Bob



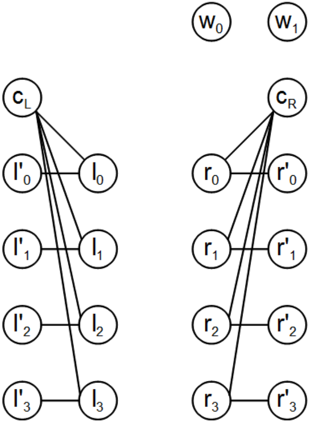
Part II: use known lower bound

$k_n := \lfloor \frac{n}{10} \rfloor$

Ex.: $n = 20, k_n = 2,$
$a = (0,0,0,1), b = (0,1,1,1)$

Alice	Bob
$L := \{l_i \| i < 2k_n\}$ $L' := \{l'_i \| i < 2k_n\}$ c_L	$R := \{r_i \| i < 2k_n\}$ $R' := \{r'_i \| i < 2k_n\}$ $c_R, W := \{w_i \| i < n - 8k_n - 2\}$
$E_A := \{\{l_i, l'_i\} \| i < 2k_n\} \cup \{\{l_i, c_L\} \| i < 2k_n\}$	$E_B := \{\{r_i, r'_i\} \| i < 2k_n\} \cup \{\{r_i, c_R\} \| i < 2k_n\}$

Alice | Bob



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$$k_n := \lfloor \frac{n}{10} \rfloor$$

Alice	Bob
$L := \{l_i \mid i < 2k_n\}$ $L' := \{l'_i \mid i < 2k_n\}$ c_L	$R := \{r_i \mid i < 2k_n\}$ $R' := \{r'_i \mid i < 2k_n\}$ $c_R, W := \{w_i \mid i < n - 8k_n - 2\}$
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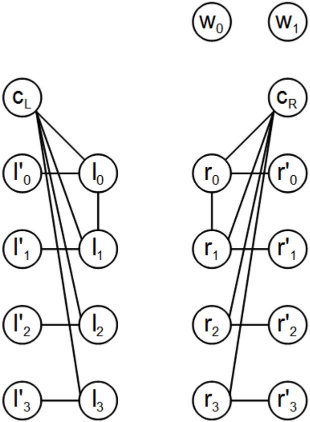

Part II: use known lower bound

$k_n := \lfloor \frac{n}{10} \rfloor$

Alice	Bob
$L := \{l_i \mid i < 2k_n\}$	$R := \{r_i \mid i < 2k_n\}$
$L' := \{l'_i \mid i < 2k_n\}$	$R' := \{r'_i \mid i < 2k_n\}$
c_L	$c_R, W := \{w_i \mid i < n - 8k_n - 2\}$
$E_A := \{\{l_i, l'_i\} \mid i < 2k_n\} \cup \{\{l_i, c_L\} \mid i < 2k_n\} \cup \{\{l_i, l_j\} \mid i < j < k_n\} \cup \{\{l_i, l_j\} \mid i > j \geq k_n\}$	$E_B := \{\{r_i, r'_i\} \mid i < 2k_n\} \cup \{\{r_i, c_R\} \mid i < 2k_n\} \cup \{\{r_i, r_j\} \mid i < j < k_n\} \cup \{\{r_i, r_j\} \mid i > j \geq k_n\}$

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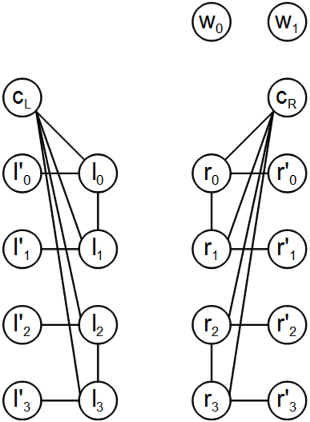
Alice | Bob



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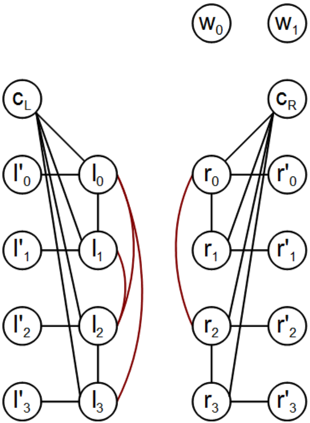
Alice Bob



Part II: use known lower bound

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Alice	Bob
$L := \{l_i \| i < 2k_n\}$	$R := \{r_i \| i < 2k_n\}$
$L' := \{l'_i \| i < 2k_n\}$	$R' := \{r'_i \| i < 2k_n\}$
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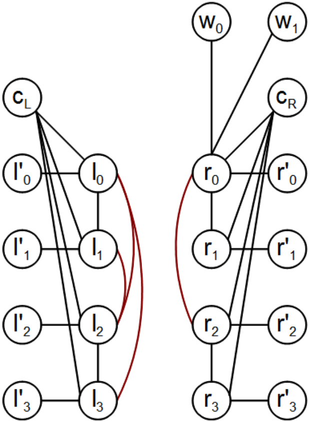
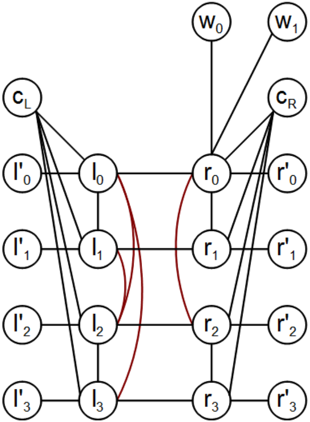
$E_A := \{\{l_i, l'_i\} \| i < 2k_n\}$ $\cup \{\{l_i, c_L\} \| i < 2k_n\}$ $\cup \{\{l_i, l_j\} \| i < j < k_n\}$ $\cup \{\{l_i, l_j\} \| i > j \geq k_n\}$ $\cup \{\{l_{i \bmod k_n}, l_{k_n + \lfloor \frac{i}{k_n} \rfloor}\} \| a_i = 0\}$	$E_B := \{\{r_i, r'_i\} \| i < 2k_n\}$ $\cup \{\{r_i, c_R\} \| i < 2k_n\}$ $\cup \{\{r_i, r_j\} \| i < j < k_n\}$ $\cup \{\{r_i, r_j\} \| i > j \geq k_n\}$ $\cup \{\{r_{i \bmod k_n}, r_{k_n + \lfloor \frac{i}{k_n} \rfloor}\} \| b_i = 0\}$ $\cup \{\{r_0, w_i\}\}$

Ex.: $n = 20, k_n = 2,$
$a = (0, 0, 0, 1), b = (0, 1, 1, 1)$

Alice | Bob



Part II: use known lower bound

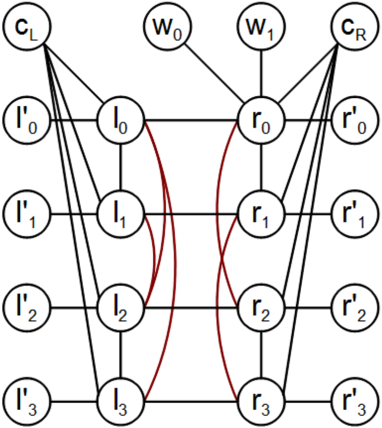
$k_n := \lfloor \frac{n}{10} \rfloor$		Ex.: $n = 20, k_n = 2$, $a = (0,0,0,1), b = (0,1,1,1)$
Alice	Bob	Alice Bob
$L := \{l_i \mid i < 2k_n\}$ $L' := \{l'_i \mid i < 2k_n\}$ c_L	$R := \{r_i \mid i < 2k_n\}$ $R' := \{r'_i \mid i < 2k_n\}$ $c_R, W := \{w_i \mid i < n - 8k_n - 2\}$	
E_A $:= \{\{l_i, l'_i\} \mid i < 2k_n\}$ $\cup\{\{l_i, c_L\} \mid i < 2k_n\}$ $\cup\{\{l_i, l_j\} \mid i < j < k_n\}$ $\cup\{\{l_i, l_j\} \mid i > j \geq k_n\}$ $\cup\{\{l_{i \bmod k_n}, l_{k_n + \lfloor \frac{i}{k_n} \rfloor}\} \mid a_i = 0\}$	E_B $:= \{\{r_i, r'_i\} \mid i < 2k_n\}$ $\cup\{\{r_i, c_R\} \mid i < 2k_n\}$ $\cup\{\{r_i, r_j\} \mid i < j < k_n\}$ $\cup\{\{r_i, r_j\} \mid i > j \geq k_n\}$ $\cup\{\{r_{i \bmod k_n}, r_{k_n + \lfloor \frac{i}{k_n} \rfloor}\} \mid b_i = 0\}$ $\cup\{\{r_0, w_i\}\}$	
$C := \{\{l_i, r_i\} \mid i < 2k_n\}$		

Part II: use known lower bound

Needed:
$\mathcal{R}_{k_n^2} : \{\text{A, B}\} \times \{0,1\}^{k_n^2} \to \{(H,C) \mid H \subseteq G, (H, G \setminus H, C) \text{ cut}\}$
such that $\text{disj}_{k_n^2}(a,b) = \text{diam}'_4(\mathcal{R}_{k_n^2}(A,a), \mathcal{R}_{k_n^2}(B,b))$

Part II: use known lower bound

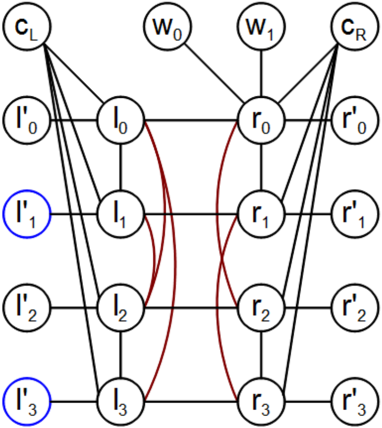
$$\mathrm{disj}_{k_n^2}(a,b)=1 \quad\Rightarrow\quad \mathrm{diam}'_4(\mathcal{R}_{k_n^2}(A,a),\mathcal{R}_{k_n^2}(B,b))=1$$



$$a=(0,0,0,1) \quad b=(0,1,1,0)$$

Part II: use known lower bound

$$\text{disj}_{k_n^2}(a,b) = 1 \quad \Rightarrow \quad \text{diam}'_4(\mathcal{R}_{k_n^2}(A,a), \mathcal{R}_{k_n^2}(B,b)) = 1$$



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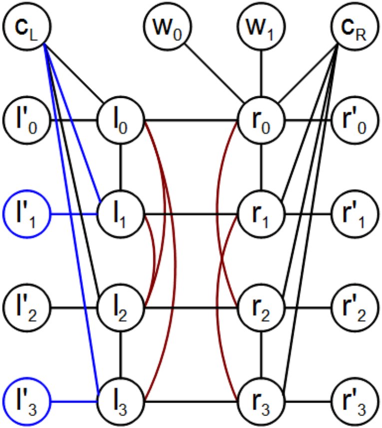

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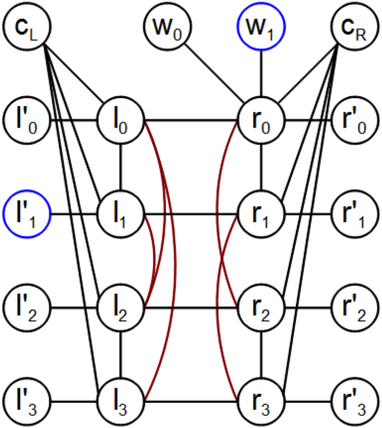

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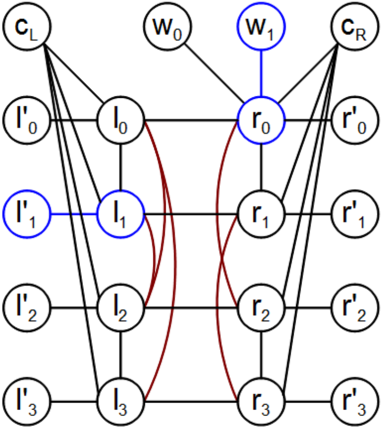

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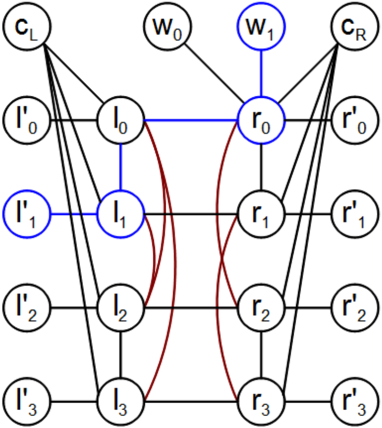

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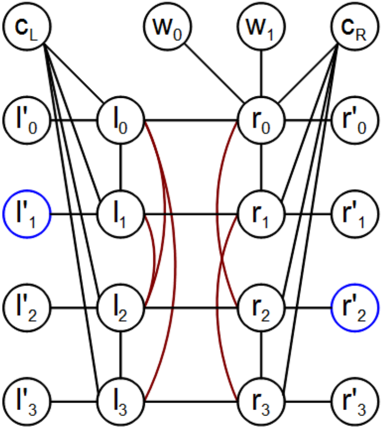

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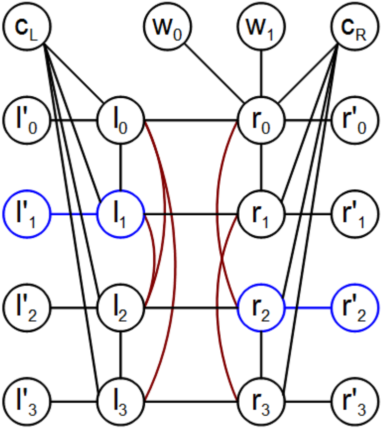

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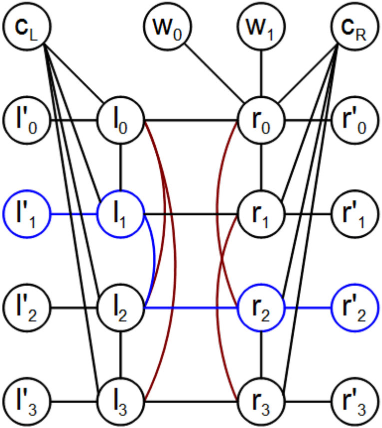

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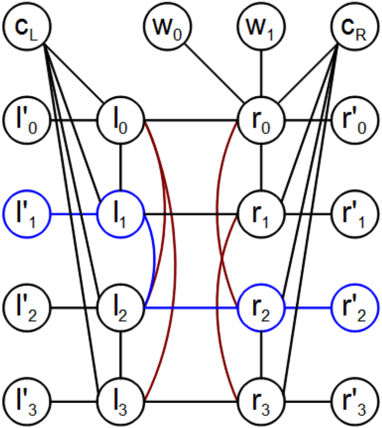

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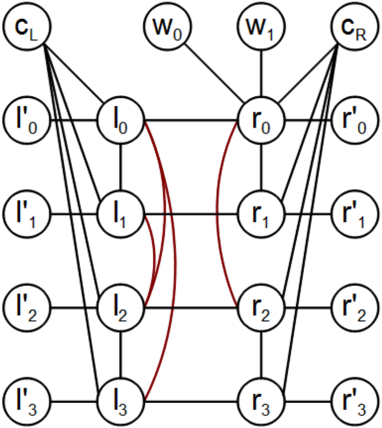

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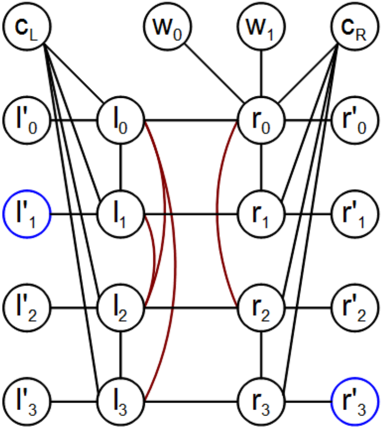

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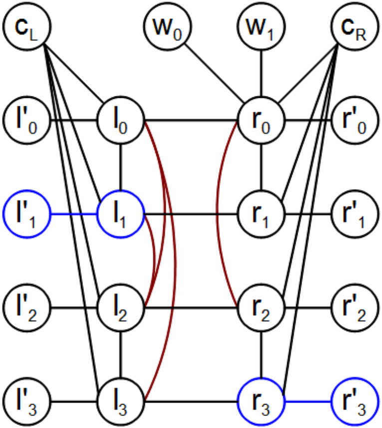

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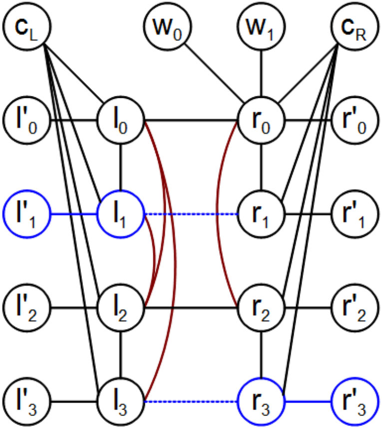

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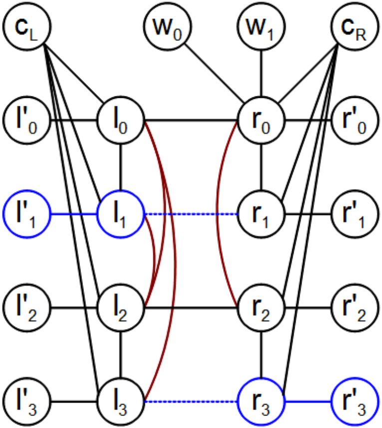

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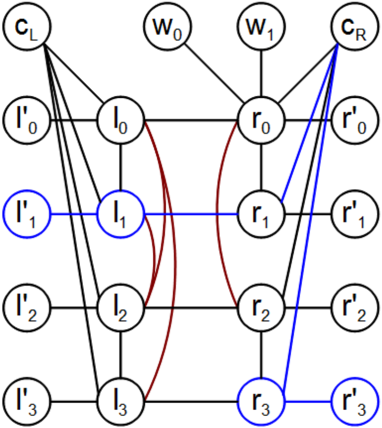
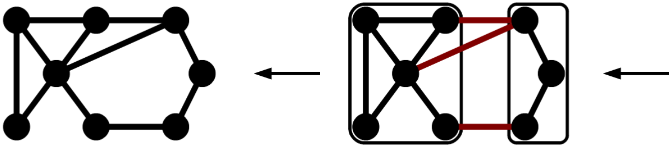


$a=(0,0,0,1) \quad b=(0,1,1,1)$

Part II: use known lower bound

Theorem

$\forall n \geq 10 \; \forall B \geq 1 \; \forall \epsilon > 0$ *sufficiently small:* $R_\epsilon^{dc}(\mathrm{diam}_4) \in \Omega(\frac{n}{B})$
(even when diameter is bounded by 5)



Alice: a
Bob: b

$\mathrm{diam}_4(G)$ $\qquad$ $\mathrm{diam}_4'((G_a, C), (G_b, C))$ $\qquad$ $\mathrm{disj}_{k_n^2}(a, b)$

$$R_\epsilon^{dc}(\mathrm{diam}_4) \geq \frac{R_\epsilon^{cc}(\mathrm{diam}_4')}{2|C|B} \geq \frac{R_\epsilon^{cc}(\mathrm{disj}_{k_n^2})}{2|C|B}$$

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$\forall n \geq 10 \; \forall B \geq 1 \; \forall \epsilon > 0$ *sufficiently small:* $R^{dc}_{\epsilon}(\text{diam}_4) \in \Omega(\frac{n}{B})$
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Lemma

$\forall \epsilon > 0$ *sufficienty small:* $R^{cc}_{\epsilon}(\text{disj}_k) \in \Omega(k)$

$k_n = \lfloor \frac{n}{10} \rfloor$, $|C| = 2k_n = 2\lfloor \frac{n}{10} \rfloor$

Part II: use known lower bound

Theorem

$\forall n \geq 10\ \forall B \geq 1\ \forall \epsilon > 0$ *sufficiently small:* $R_\epsilon^{dc}(\mathrm{diam}_4) \in \Omega(\frac{n}{B})$
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Lemma

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$k_n = \lfloor \frac{n}{10} \rfloor$, $|C| = 2k_n = 2\lfloor \frac{n}{10} \rfloor$

$$\Rightarrow R_\epsilon^{dc}(\mathrm{diam}_4) \geq \frac{R_\epsilon^{cc}(\mathrm{disj}_{k_n^2})}{2|C|B} \in \Omega\left(\frac{n}{B}\right)$$

□

Other lower bounds using this technique

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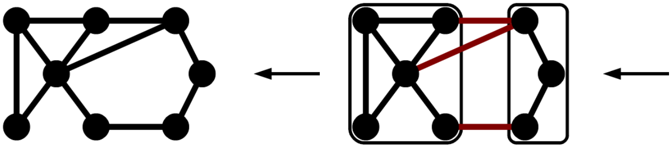

Alice: a
Bob: b

$f(G)$ $\quad f'((G_a, C), (G_b, C))$ $\quad \mathrm{disj}_{k_n^2}(a, b)$

$$R_\epsilon^{dc}(f) \geq \frac{R_\epsilon^{cc}(f')}{2|C|B} \geq \frac{R_\epsilon^{cc}(\mathrm{disj}_{k_n^2})}{2|C|B}$$

Other lower bounds using this technique

Theorem

$\forall \delta > 0 \, \forall n \geq 16\lceil \frac{3}{4\delta} \rceil + 8 \, \forall B \geq 1 \, \forall \epsilon > 0$ *sufficiently small: any distributed* ϵ*-error algorithm that* $(\frac{3}{2} - \delta)$*-approximates the diameter of a graph needs* $\Omega(\frac{\sqrt{\delta n}}{B})$ *rounds.*

Theorem

$\forall \delta > 0 \, \forall n \geq 16\lceil \frac{2}{\delta} \rceil + 4 \, \forall B \geq 1 \, \forall \epsilon > 0$ *sufficiently small: any distributed* ϵ*-error algorithm that* $(2 - \delta)$*-approximates the girth of a graph needs* $\Omega(\frac{\sqrt{\delta n}}{B})$ *rounds.*