The Adjoint of a Polytope

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joint works with Kristian Ranestad (Universitetet i Oslo) / Boris Shapiro (Stockholms universitet) & Bernd Sturmfels (MPI MiS Leipzig / UC Berkeley)
The Adjoint of a Polygon

Wachspress (1975)

Definition

The adjoint $A_P$ of a polygon $P \subset \mathbb{R}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of $P$.

$\deg A_P = |V(P)| - 3$
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Generalization to higher-dimensional polytopes?
The Adjoint of a Polytope

Warren (1996)

- $P$: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of $P$
- $\tau(P)$: triangulation of $P$ using only the vertices of $P$

Definition

adj \( \tau(P) \) \( (t) \) := \[
\sum_{\sigma \in \tau(P)} \text{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),
\]

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

Theorem (Warren)

I adj \( \tau(P) \) \( (t) \) is independent of the triangulation \( \tau(P) \). So adj \( P \) := adj \( \tau(P) \).

II If $P$ is a polygon, then $Z(\text{adj} P) = A P^\ast$.

(Recall: $P^\ast$ = \{ $x \in \mathbb{R}^n | \forall v \in V(P)$ : $\ell_v(x) \geq 0$ \} dual polytope of $P$)

Geometric definition using a vanishing condition `a la Wachspress

II - XVIII
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$$\text{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \text{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$. 

Theorem (Warren)

$\text{adj}_{\tau(P)}(t)$ is independent of the triangulation $\tau(P)$. So $\text{adj}_P := \text{adj}_{\tau(P)}(P)$. 

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If $P$ is a polygon, then $Z(\text{adj} P) = A_P^*$.

(Recall: $P^* = \{x \in \mathbb{R}^n | \forall v \in V(P) : \ell_v(x) \geq 0\}$ dual polytope of $P$)
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Theorem (K., Ranestad)
If $\mathcal{H}_P$ is simple (i.e. through any point in $P$ pass $\leq n$ hyperplanes), there is a unique hypersurface $A_P$ in $\mathbb{P}^n$ of degree $d - n - 1$ passing through $\mathcal{R}_P$. $A_P$ is called the adjoint of $P$. 

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adjoint quadric surface

adjoint plane
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**Proposition (K., Ranestad)**

Warren’s adjoint polynomial $\text{adj} P$ vanishes along $\mathcal{R}_P^*$. If $\mathcal{H}_P^*$ is simple, then $Z(\text{adj} P) = A_P^*$. 

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IV - XVIII
Application 1: Segre Classes of Monomial Schemes

Aluffi

- $V$: smooth variety
- $X_1, \ldots, X_n$: smooth hypersurfaces meeting with normal crossings in $V$
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- $\mathcal{A} \subset \mathbb{Z}^n_{\geq 0}$ defines a **monomial subscheme**

$S_\mathcal{A} = \bigcap_{\mathcal{I} \in \mathcal{A}} X^\mathcal{I}$ and a Newton region $N_\mathcal{A} \subset \mathbb{R}^n_{\geq 0}$

**Example:** $n = 2$

$\mathcal{A} = \{(2, 6), (3, 4), (4, 3), (5, 1), (7, 0)\}$
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$$\mathcal{N}_\mathcal{A} := \mathbb{R}^n_{\geq 0} \setminus \text{convHull} \left( \bigcup_{\mathcal{I} \in \mathcal{A}} (\mathbb{R}^n_{\geq 0} + \mathcal{I}) \right)$$

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**Theorem (Aluffi, (K., Ranestad))**

The Segre class of $S_A$ in the Chow ring of $V$ is

$$n! \prod_{v \in V(N_A)} \ell_v(-X) \left( -X \right)^{\text{adj}_{N_A}} / \prod_{v \in V(N_A)} \ell_v(-X),$$

if $N_A$ is finite.

**Example:** $n = 2$

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Application 1: Segre Classes of Monomial Schemes

- \( N_A \) may have vertices at \( \infty \) in the direction of the standard basis vectors \( e_1, \ldots, e_n \)

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- for vertex $v_i$ at $\infty$ in direction of $e_i$:
  \[ \ell_{v_i}(t) := -t_i \]

**Theorem (Aluffi, (K., Ranestad))**

The Segre class of $S_A$ in the Chow ring of $V$ is

\[
\frac{n! X_1 \cdots X_n \text{adj}_{N_A}(-X)}{\prod_{v \in V(N_A)} \ell_v(-X)}.
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Example: $2X_1X_2 \, \text{adj}_{N_A}(-X_1, -X_2)$

$$\frac{X_2(1 + 2X_1 + 6X_2)(1 + 3X_1 + 4X_2)(1 + 5X_1 + X_2)(1 + 7X_1)}{1 - 15t_1 - 22t_2 + 71t_1^2 + 212t_1t_2 + 95t_2^2 - 105t_1^3 - 476t_1^2t_2 - 511t_1t_2^2 - 84t_2^3},$$

where

$$\text{adj}_{N_A}(t) = 1 - 15t_1 - 22t_2 + 71t_1^2 + 212t_1t_2 + 95t_2^2 - 105t_1^3 - 476t_1^2t_2 - 511t_1t_2^2 - 84t_2^3.$$
Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

- $P$: convex polytope in $\mathbb{R}^n$
- $\mu_P$: uniform probability distribution on $P$

$$m_I(P) := \int_{\mathbb{R}^n} w_{i_1} w_{i_2} \ldots w_{i_n} d\mu_P$$

Proposition (K., Shapiro, Sturmfels)

$$\sum_{I \in \mathbb{Z}^n_{\geq 0}} c_I m_I(P) t_I = \text{adj} P(t) \cdot \text{vol}(P) \prod_{v \in V(P)} \ell_v(t),$$

where $c_I := (i_1 + i_2 + \ldots + i_n + n, i_1, i_2, \ldots, i_n, n)$.  

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- moments

$$m_{\mathcal{I}}(P) := \int_{\mathbb{R}^n} w_1^{i_1} w_2^{i_2} \ldots w_n^{i_n} d\mu_P \quad \text{for } \mathcal{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_{\geq 0}^n$$
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Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}_{\geq 0}^n} c_{\mathcal{I}} m_{\mathcal{I}}(P) t^{\mathcal{I}} = \frac{\text{adj}_P(t)}{\text{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

where $c_{\mathcal{I}} := (i_1 + i_2 + \ldots + i_n + n)_{i_1, i_2, \ldots, i_n, n}$. 
Application 3: Barycentric Coordinates

\[ \beta_{v_i}(p) := \frac{\text{area}(\triangle_i)}{\text{area}(\triangle_1) + \text{area}(\triangle_2) + \text{area}(\triangle_3)} \]

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**Definition**
Let \( P \) be a convex polytope in \( \mathbb{R}^n \). A set of functions \( \{ \beta_u : P^\circ \to \mathbb{R} \mid u \in V(P) \} \) is called **generalized barycentric coordinates** for \( P \) if, for all \( p \in P^\circ \),

(i) \( \forall u \in V(P) : \beta_u(p) > 0, \)

(ii) \[ \sum_{u \in V(P)} \beta_u(p) = 1, \] and

(iii) \[ \sum_{u \in V(P)} \beta_u(p)u = p. \]
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(iii) \( \sum_{u \in V(P)} \beta_u(p) u = p. \)

Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!
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Examples of generalized barycentric coordinates for arbitrary polytopes:

- mean value coordinates
- Wachspress coordinates

Applications of generalized barycentric coordinates include:
- mesh parameterizations in geometric modelling
- deformations in computer graphics
- polyhedral finite element methods

The Wachspress coordinates are the unique generalized barycentric coordinates which are rational functions of minimal degree.
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$V(P) \leftrightarrow_{1:1} \mathcal{F}(P^*)$

$v \mapsto F_v$
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Definition (Warren)

The Wachspress coordinates of $P$ are:

$$\beta_u(t) := \text{adj}_{F_u(t)} \prod_{F \in \mathcal{F}(P)} u \notin F \ell_v F(t).$$
Wachspress Coordinates

Warren (1996)

- $P$: convex polytope in $\mathbb{R}^n$
- $\mathcal{F}(P)$: set of facets of $P$

$$V(P) \xleftrightarrow{1:1} \mathcal{F}(P^*)$$

$v \mapsto F_v$

$\text{Definition (Warren)}$

The Wachspress coordinates of $P$ are

$$\forall u \in V(P) : \beta_u(t) := \frac{\text{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P) : u \notin F} \ell_{v_F}(t)}{\text{adj}_{P^*}(t)}.$$
∀ \( u \in V(P) \): \( \beta_u(t) := \frac{\text{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P): u \notin F} \ell_{v_F}(t)}{\text{adj}_{P^*}(t)} \).

- \( P \): polytope in \( \mathbb{P}^n \) with \( d \) facets
- \( \mathcal{H}_P \): simple hyperplane arrangement spanned by facets of \( P \)
Wachspress Map

\[ \forall u \in V(P) : \quad \beta_u(t) := \frac{\text{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P) : u \notin F} \ell_{v_F}(t)}{\text{adj}_{P^*}(t)}. \]

- \( P \): polytope in \( \mathbb{P}^n \) with \( d \) facets
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The numerators of the Wachspress coordinates define the **Wachspress map**: 

\[ \omega_P : \mathbb{P}^n \rightarrow \mathbb{P}^{|V(P)|-1}, \]

\[ t \mapsto \]
Wachspress Map

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The numerators of the Wachspress coordinates define the \textbf{Wachspress map}:

\[ \omega_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{|V(P)| - 1}, \]

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where \( \ell_F \) is a homogeneous linear equation defining the hyperplane span \{\( F \)\}. 

\[ \text{XI - XVIII} \]
Wachspress Map

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$
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\[ t \mapsto \left( \prod_{F \in \mathcal{F}(P) : u \notin F} \ell_F(t) \right) \quad u \in V(P) \]

Theorem (K., Ranestad)

The base locus of the Wachspress map $\omega_P$ is the residual arrangement $R_P$. 
Wachspress Map

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$$
\begin{pmatrix}
\prod_{F \in \mathcal{F}(P): u \notin F} \ell_F(t)
\end{pmatrix}_{u \in V(P)}
$$

Theorem (K., Ranestad)
The base locus of the Wachspress map $\omega_P$ is the residual arrangement $\mathcal{R}_P$.

$$
\forall u \in V(P) : \omega_P, u \in \Omega_P := H^0(\mathbb{P}^n, I_{\mathcal{R}_P}(d - n))
$$
Wachspress Map

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets
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\Rightarrow \forall u \in V(P) : \omega_{P,u} \in \Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d - n))
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Theorem (K., Ranestad)

$$
\dim \Omega_P = \vert V(P) \vert, \text{ so } \{ \omega_{P,u} \mid u \in V(P) \} \text{ is a basis of } \Omega_P.
$$
**Wachspress Map**

- \( P \): polytope in \( \mathbb{P}^n \) with \( d \) facets
- \( \mathcal{H}_P \): simple hyperplane arrangement spanned by facets of \( P \)
- Wachspress map: \( \omega_P : \mathbb{P}^n \rightarrow \mathbb{P}^{\left| V(P) \right| - 1} \)

\[ t \mapsto \left( \prod_{F \in \mathcal{F}(P) : u \notin F} \ell_F(t) \right) \]

\( u \in V(P) \)

**Theorem (K., Ranestad)**

The base locus of the Wachspress map \( \omega_P \) is the residual arrangement \( \mathcal{R}_P \).

\[ \Rightarrow \forall u \in V(P) : \omega_P, u \in \Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d - n)) \]

**Theorem (K., Ranestad)**

\( \dim \Omega_P = |V(P)| \), so \( \{ \omega_P, u \mid u \in V(P) \} \) is a basis of \( \Omega_P \).

\[ \Rightarrow \omega_P : \mathbb{P}^n \rightarrow \mathbb{P}(\Omega_P^*) \cong \mathbb{P}^{\left| V(P) \right| - 1} \]
Wachspress Map

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$
- $\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d - n))$
- $W_P := \omega_P(\mathbb{P}^n)$ is the Wachspress variety
Wachspress Map

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- \( V_P := \text{span}\{\omega_P(A_P)\} \)

Theorem (K., Ranestad)

\[ \dim V_P = |V_P(P)| - n - 2. \]

The projection \( \text{pr}_{V_P} : \mathbb{P}(\Omega^*_P) \rightarrow \mathbb{P}^n \) restricted to the Wachspress variety \( W_P \) is the inverse of the Wachspress map \( \omega_P \).
Wachspress Map

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$
- $\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{R_P}(d - n))$
- $W_P := \omega_P(\mathbb{P}^n)$ is the Wachspress variety
- $V_P := \text{span}\{\omega_P(A_P)\}$

**Theorem (K., Ranestad)**

$$\dim V_P = |V(P)| - n - 2.$$  

The projection $\text{pr}_{V_P} : \mathbb{P}(\Omega_P^*) \rightarrow \mathbb{P}^n$ from $V_P$.
Wachspress Map

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$
- $\Omega_P := H^0(\mathbb{P}^n, \mathcal{I}_{\mathcal{R}_P}(d - n))$
- $W_P := \omega_P(\mathbb{P}^n)$ is the **Wachspress variety**
- $\mathcal{V}_P := \text{span}\{\omega_P(A_P)\}$

**Theorem (K., Ranestad)**

$\dim \mathcal{V}_P = |V(P)| - n - 2$.

The projection $\text{pr}_{\mathcal{V}_P} : \mathbb{P}(\Omega_P^*) \to \mathbb{P}^n$ from $\mathcal{V}_P$ restricted to the Wachspress variety $W_P$ is the inverse of the Wachspress map $\omega_P$.  

![Diagram](image-url)
Theorem (Irving, Schenck)
Let $P$ be a $d$-gon in $\mathbb{P}^2$. 

Let $P$ be a $d$-gon in $\mathbb{P}^2$. 

\[ \mathbb{P}^2 \xrightarrow{\text{pr}_{\mathbb{V}_P}} \mathbb{P}(\Omega^*_P) \cong \mathbb{P}^{d-1} \]

\[ \mathbb{P}^2 \xleftarrow{\text{pr}_{\mathbb{V}_P}} \mathbb{P}(\Omega^*_P) \cong \mathbb{P}^{d-4} \]
Theorem (Irving, Schenck)

Let $P$ be a $d$-gon in $\mathbb{P}^2$.

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Theorem (Irving, Schenck)

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- If $d = 4$, the image of the adjoint line $A_P$ is a point.
Wachspress Threefolds

$\omega_P(\Pi_{\triangle_1 : \Pi_{\triangle_2}}) \otimes (\Pi_{\square_1 : \Pi_{\square_2 : \Pi_{\square_3}}})$  

$W_P P_1 \times P_2 \hookrightarrow P_5$  

$\mathcal{R}_P$  

$\mathcal{A}_P$  

adjoint plane

projection from point contracts ruling of lines

$\omega_P |_{\mathcal{A}_P}$  

line twisted cubic curve
Wachspress Threefolds

\[ \omega_P \mid A_P \text{ projection from point} \]

contracts ruling of lines

\[ \omega_P \mid (\ell_{\Delta_1} : \ell_{\Delta_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3}) \]
Wachspress Threefolds

$P$

$\Delta_1 \cap \Delta_2$

$\mathcal{R}_P$

$A_P$

$\omega_P$

$\omega_P|_{AP}$ projection from point contracts ruling of lines

$\omega_P (\ell_{\Delta_1} : \ell_{\Delta_2}) \otimes (\ell_{\Box_1} : \ell_{\Box_2} : \ell_{\Box_3})$

$W_P$

$\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$
Wachspress Threefolds

\[ \mathbb{P} \]

\[ \mathbb{R}_P \]

\[ A_P \]

adjoint plane

\[ \omega_P \]

\[ \left( \ell_{\Delta_1} : \ell_{\Delta_2} \right) \otimes \left( \ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3} \right) \]

\[ \mathbb{W}_P \]

\[ \mathbb{W}_P |_{A_P} \]

projection from point

\[ \omega_P |_{A_P} \]
Wachspress Threefolds

\[ \mathcal{R}_P \]

adjoint plane

\[ \omega_P \]

(line) twisted cubic curve

Projection from point

\[ W_P \]

line
Wachspress Threefolds

\( P \)

\( \square_1 \cap \square_2 \cap \square_3 \)

\( \Delta_1 \cap \Delta_2 \)

\( R_P \)

\( A_P \), adjoint plane

\( \omega_P \)

\( (\ell_{\Delta_1} : \ell_{\Delta_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3}) \)

\( W_P \)

\( \omega_P|_{A_P} \), projection from point

\( \omega_P(A_P) \), line

adjoint quadric surface

\( X V - X V I I I \)
Wachspress Threefolds

\[ P \]

\[ \square_1 \cap \square_2 \cap \square_3 \]

\[ \Delta_1 \cap \Delta_2 \]

\[ R_P \]
adjoint plane

\[ \omega_P \]
line

\[ (\ell_{\Delta_1} : \ell_{\Delta_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3}) \]

\[ W_P \]
projection from point

\[ \omega_P|_{A_P} \]
line

\[ \omega_P(A_P) \]
adjoint quadric surface

\[ (\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4) \]
Wachspress Threefolds

\[ \mathcal{R}_P \]

adjoint plane

\[ \mathcal{A}_P \]

line

\[ \omega_P \]

projection from point

\[ \omega_P(A_P) \]

adjoint quadric surface

\[ \omega_P|_{A_P} \]

line

\[ (l_1 : l_2) \otimes (l_3 : l_4) \]

\[ (l_5 : l_6) \otimes (l_7 : l_8) \]

\[ \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \]

\[ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7 \]
Wachspress Threefolds

$\mathcal{R}_P$

$\mathcal{A}_P$ adjoint plane

$\omega_P \big( \ell_1 : \ell_2 \big) \otimes \big( \ell_3 : \ell_4 \big)$

$\mathcal{W}_P$ projection from point

$\omega_P|_{\mathcal{A}_P}$ line

$\omega_P(\mathcal{A}_P)$

$\square_1 \cap \square_2 \cap \square_3 \cap \Delta_1 \cap \Delta_2$

adjoint quadric surface

$(\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4)$

contracts ruling of lines
Wachspress Threefolds

\( P \)

\[ \square_1 \cap \square_2 \cap \square_3 \]

\( \Delta_1 \cap \Delta_2 \)

\( R_P \)

adjoint plane

\( \omega_P \)

\( (\ell_{\Delta_1} : \ell_{\Delta_2}) \otimes (\ell_{\square_1} : \ell_{\square_2} : \ell_{\square_3}) \)

\( W_P \)

\( \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \)

\( \omega_P|_{A_P} \)

projection from point

\( \omega_P(A_P) \)

line

\( \text{adjoint quadric surface} \)

\( (\ell_1 : \ell_6) \otimes (\ell_2 : \ell_5) \otimes (\ell_3 : \ell_4) \)

\( \text{contracts ruling of lines} \)

\( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7 \)

\( \text{twisted cubic curve} \)
Wachspress Threefolds

- $P$: polytope in $\mathbb{P}^3$ with $d$ facets
- $H_P$: simple hyperplane arrangement spanned by facets of $P$
- $a$: number of isolated points in $R_P$
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Wachspress Threefolds

- \( P \): polytope in \( \mathbb{P}^3 \) with \( d \) facets
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**Proposition (K., Ranestad)**

The Wachspress variety \( W_P \subset \mathbb{P}^{2d-5} \) is a threefold of degree

\[
2b + 4c - a - \frac{1}{2} (d - 3)(d^2 - 11d + 26) = b + 2c + 1 - \frac{1}{6} (d - 3)(d - 4)(d - 11)
\]

and sectional genus \( b + 2c + 1 + \frac{1}{2} (d - 3)(d - 6) \).
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The image of the adjoint surface $A_P$ under $\omega_P$ is a surface iff $P$ is neither a tetrahedron, a triangular prism nor a cube.
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The image of the adjoint surface $A_P$ under $\omega_P$ is a surface iff $P$ is neither a tetrahedron, a triangular prism nor a cube. In that case, its degree is

$$2b + 4c - a - \frac{1}{2} (d - 3)(d - 4)(d - 6) = b + 2c + 1 - \frac{1}{6} (d - 3)(d^2 - 12d + 38)$$

and its sectional genus is $b + 2c + 1 - \frac{1}{2} (d - 3)(d - 4)$. 
Why “Adjoint”? 

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$

Idea:

$$P \rightarrow \mathcal{H}_P$$

hypersurface of degree $d$
Why “Adjoint”? 

- \( P \): polytope in \( \mathbb{P}^n \) with \( d \) facets
- \( \mathcal{H}_P \): simple hyperplane arrangement spanned by facets of \( P \)
- \( \mathcal{R}_P^c \): codimension-\( c \) part of \( \mathcal{R}_P \)

Idea:

\[
P \quad \longrightarrow \quad \mathcal{H}_P \quad \longrightarrow \quad D
\]

hypersurface of degree \( d \)  

**polytopal hypersurface:**  
hypersurface of degree \( d \), multiplicity \( c \) along \( \mathcal{R}_P^c \), smooth outside of \( \mathcal{R}_P \)
Why “Adjoint”? 

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets 
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$ 
- $\mathcal{R}_P^c$: codimension-$c$ part of $\mathcal{R}_P$

Idea: 

$$\xymatrix{ & \mathbb{P}^n \ar[dl] \ar[rr]^{\text{blowup } \pi} && X \ar[ul] & \text{smooth} \\
\mathcal{H}_P \ar[rr] & & D \ar[ll] & \tilde{D} \ar[ll] & \text{smooth} \\
P \ar[rr] & & \text{hypersurface of degree } d \text{, multiplicity } c \text{ along } \mathcal{R}_P^c, \text{ smooth outside of } \mathcal{R}_P}$

Def.: An adjoint to $\tilde{D}$ in $X$ is a hypersurface $A$ in $X$ s.t. $[A] = K_X + [\tilde{D}]$.

Proposition (K., Ranestad): $\tilde{D}$ has a unique adjoint $A$ in $X$, and thus a unique canonical divisor: $A \cap \tilde{D}$. Moreover, $\pi(A) = A \cap \mathcal{P}$. When can we find a polytopal hypersurface $D$?
Why “Adjoint”? 

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Idea: 

\[
\begin{array}{c}\mathbb{P}^n \quad \xleftarrow{\text{blowup } \pi} \quad X \quad \text{smooth} \\
\uparrow \quad \uparrow \\
P \xrightarrow{\text{hypersurface of degree } d} \mathcal{H}_P \xrightarrow{\text{polytopal hypersurface: hypersurface of degree } d, \text{ multiplicity } c \text{ along } \mathcal{R}_P^c, \text{ smooth outside of } \mathcal{R}_P} D \xrightarrow{\tilde{D} \text{ smooth}} \end{array}
\]

Adjunction formula: 

\[
K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}
\]
Why “Adjoint”? 

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets 
- $\mathcal{H}_P$: simple hyperplane arrangement spanned by facets of $P$ 
- $\mathcal{R}_P^c$: codimension-$c$ part of $\mathcal{R}_P$ 

Idea: 

\[ \mathbb{P}^n \xrightarrow{\text{blowup } \pi} X \text{ smooth} \]

\[ P \xrightarrow{\text{hypersurface of degree } d} \mathcal{H}_P \xrightarrow{\text{polytopal hypersurface:}} D \xrightarrow{\text{hypersurface of degree } d, \text{ multiplicity } c \text{ along } \mathcal{R}_P^c, \text{ smooth outside of } \mathcal{R}_P} \tilde{D} \text{ smooth} \]

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Why “Adjoint”?  

- $P$: polytope in $\mathbb{P}^n$ with $d$ facets  
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Idea:  

\[ \mathbb{P}^n \xrightarrow{\text{blowup } \pi} X \quad \text{smooth} \]

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Adjunction formula:  

\[ K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}} \]

Def.: An **adjoint to $\tilde{D}$ in $X$** is a hypersurface $A$ in $X$ s.t. $[A] = K_X + [\tilde{D}]$.  

Proposition (K., Ranestad)  

$\tilde{D}$ has a unique adjoint $A$ in $X$, and thus a unique canonical divisor: $A \cap \tilde{D}$. Moreover, $\pi(A) = A_P$.  

When can we find a polytopal hypersurface $D$?
Why “Adjoint”? 

- \( P \): polytope in \( \mathbb{P}^n \) with \( d \) facets
- \( \mathcal{H}_P \): simple hyperplane arrangement spanned by facets of \( P \)
- \( \mathcal{R}_P^c \): codimension-\( c \) part of \( \mathcal{R}_P \)

Idea: 

\[
\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{\text{blowup } \pi} & X \text{ smooth} \\
\uparrow & & \uparrow \\
\mathcal{H}_P & \xrightarrow{D} & \tilde{D} \text{ smooth} \\
\end{array}
\]

\( D \): polytopal hypersurface: hypersurface of degree \( d \), multiplicity \( c \) along \( \mathcal{R}_P^c \), smooth outside of \( \mathcal{R}_P \)

Adjunction formula: 
\[
K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}
\]

Def.: An \textbf{adjoint to} \( \tilde{D} \) \textbf{in} \( X \) is a hypersurface \( A \) in \( X \) s.t. \( [A] = K_X + [\tilde{D}] \).

Proposition (K., Ranestad) 

\( \tilde{D} \) has a unique adjoint \( A \) in \( X \), and thus a unique canonical divisor: \( A \cap \tilde{D} \). Moreover, \( \pi(A) = A_P \).
Proposition (K., Ranestad)

Let $P$ be a general $d$-gon in $\mathbb{P}^2$. There is a polygonal curve $D$ iff $d \leq 6$. In that case, $D$ is an elliptic curve.
Proposition (K., Ranestad)
Let $P$ be a general $d$-gon in $\mathbb{P}^2$. There is a polygonal curve $D$ iff $d \leq 6$. In that case, $D$ is an elliptic curve.

Theorem (K., Ranestad)
Let $C$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let $P$ be a general polytope of type $C$. There is a polytopal surface $D$ iff $C$ is one of:

In that case, the general $D$ is either an elliptic surface or a K3-surface.
| comb. type | facet sizes | $\mathcal{R}_P$ | $\langle a, b, c \rangle$ | $W_P$  
(dg., sec. genus) | $\overline{w}_P(A_P)$  
(dg., sec. genus) | dim $\Gamma_P$ | $\overline{w}_P(D)$  
(dg., sec. genus) |
|---|---|---|---|---|---|---|---|
| 3333 | | (0, 0, 0) | $\mathbb{P}^3$  
(1, 0) | 0 | 34 | minimal K3  
(small quartic in $\mathbb{P}^3$) |
| 44433 | | (1, 0, 0) | $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$  
(3, 0) | line | 23 | minimal K3  
(8, 5) |
| 444444 | | (0, 0, 0) | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$  
(6, 1) | twisted cubic curve | 26 | minimal K3  
(12, 7) |
| 554433 | | (2, 2, 0) | $W_P \subset \mathbb{P}^7$  
(8, 3) | quadric surface  
(2, 0) | 17 | non-minimal K3  
(14, 9) |
| 5554443 | | (1, 6, 0) | $W_P \subset \mathbb{P}^9$  
(15, 9) | del Pezzo surface in $\mathbb{P}^5$  
(5, 1) | 7 | non-minimal K3  
(19, 12) |
| 5544444 | | (0, 5, 0) | Fano 3-fold in $\mathbb{P}^9$  
(14, 8) | rational scroll in $\mathbb{P}^5$  
(4, 0) | 12 | non-minimal K3  
(18, 11) |
| 6644433 | | (3, 6, 1) | $W_P \subset \mathbb{P}^9$  
(17, 11) | rational elliptic surface in $\mathbb{P}^5$  
(7, 3) | 4 | minimal elliptic  
(22, 15) |
| 66444444 | | (0, 12, 2) | $W_P \subset \mathbb{P}^{11}$  
(27, 22) | elliptic K3-surface in $\mathbb{P}^7$  
(12, 7) | 3 | minimal elliptic  
(26, 17) |
| 55554444 | | (0, 16, 0) | $W_P \subset \mathbb{P}^{11}$  
(27, 22) | K3-surface in $\mathbb{P}^7$  
(12, 7) | 1 | non-minimal K3  
(24, 15) |