The Complex of Non-Chromatic Scales

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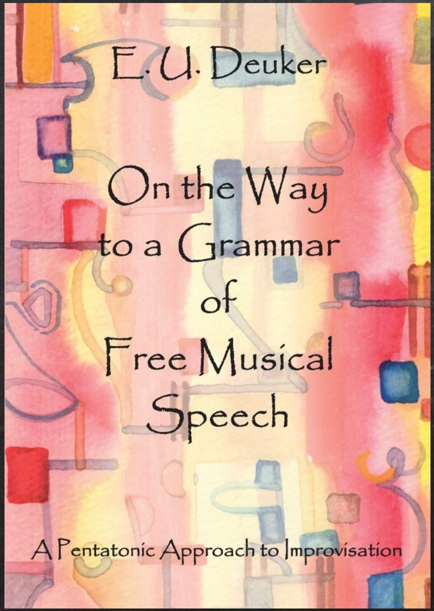


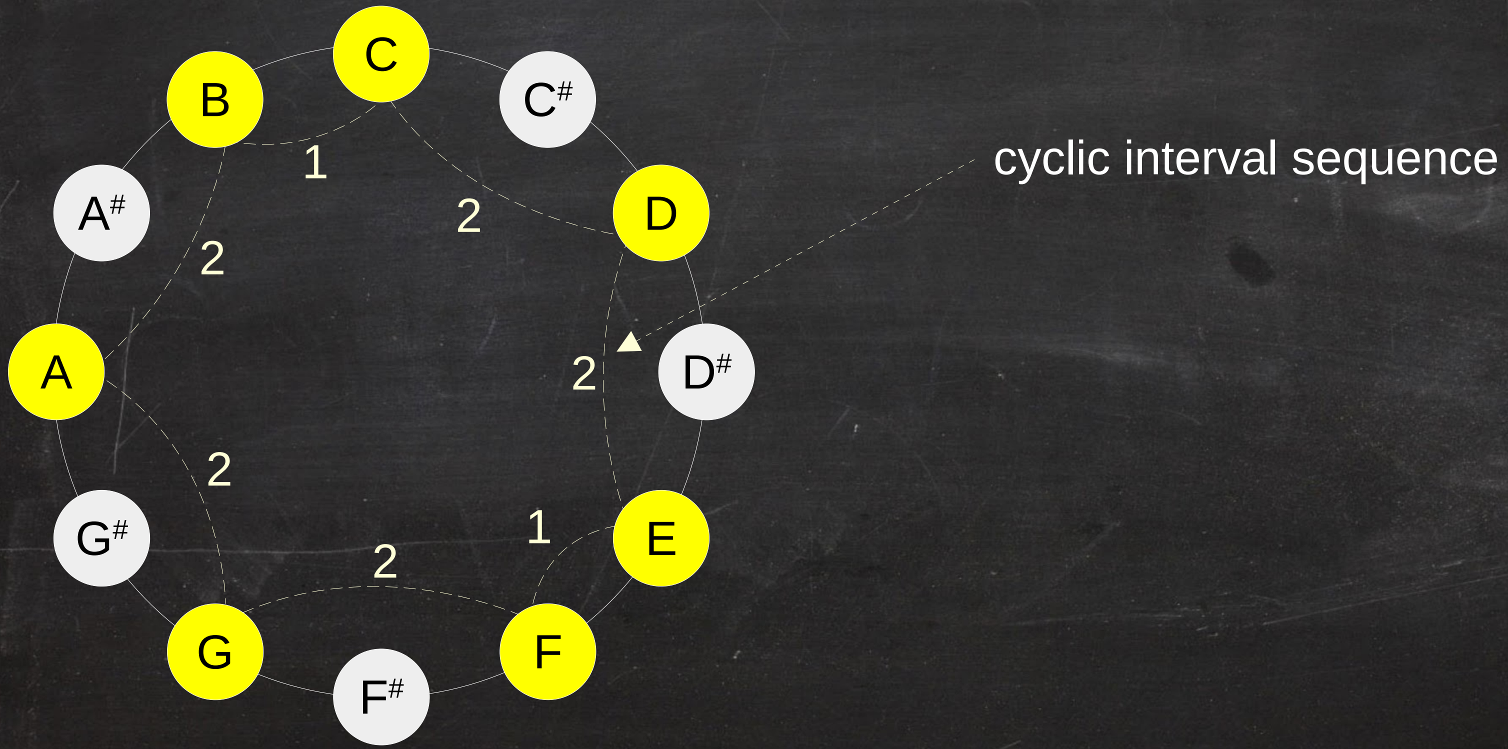
tonal musical language as a tool for free musical speech

intuitive starting points
for the spontaneous inventor of melodies (improviser)

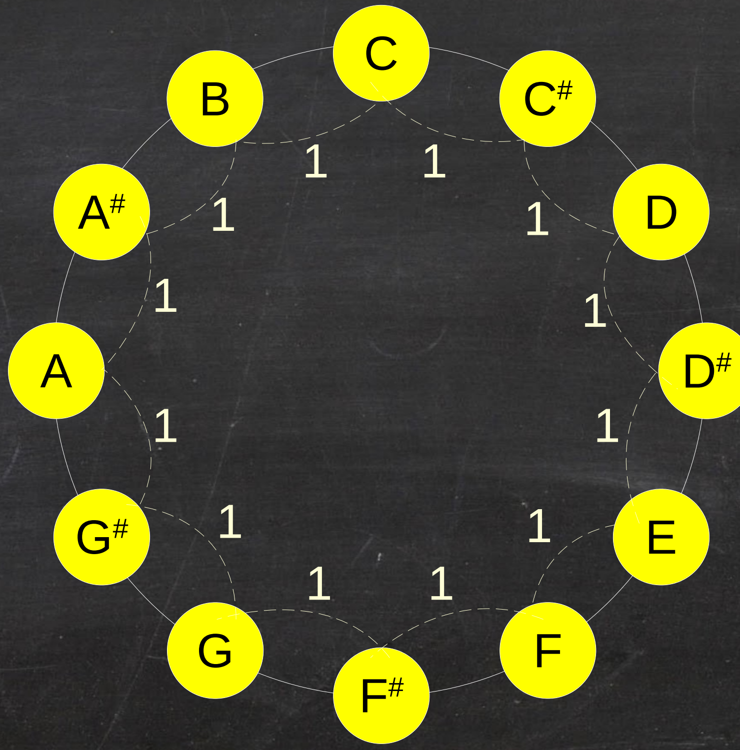
Language	Painting	Improvising
26 letters	3 primary colors (+ black / white)	7 types of non-chromatic scales
nouns, verbs, adjectives, ...	secondary colors	partly chromatic scales
sentences	paintings	melodies

cyclic scales

A **scale** is a subset of the cyclically ordered set {C,C#,D,D#,E,F,F#,G,G#,A,A#,B}.

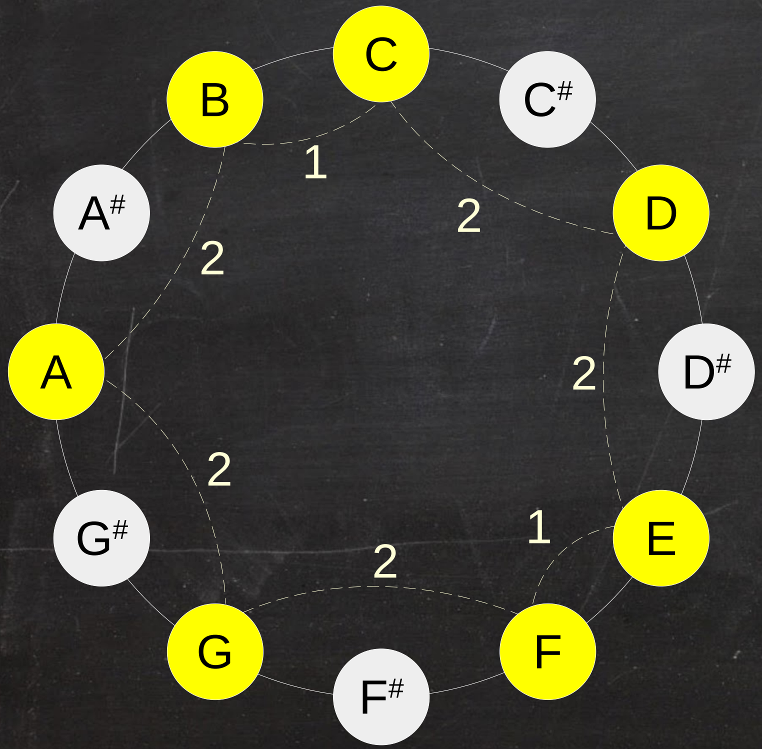


chromatic scale

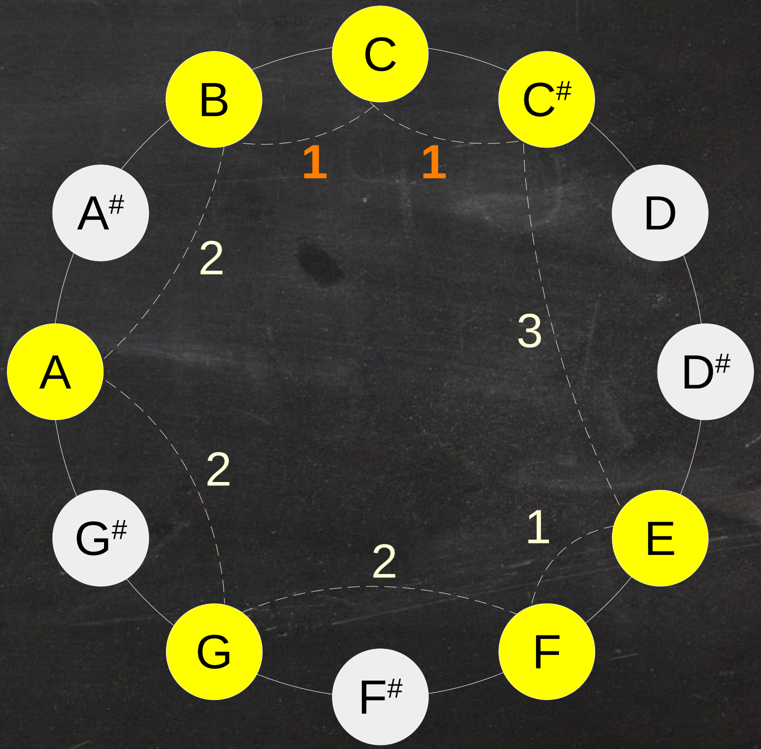


non-chromatic scales

A scale is called **non-chromatic** if its interval sequence does not contain two consecutive semitones.



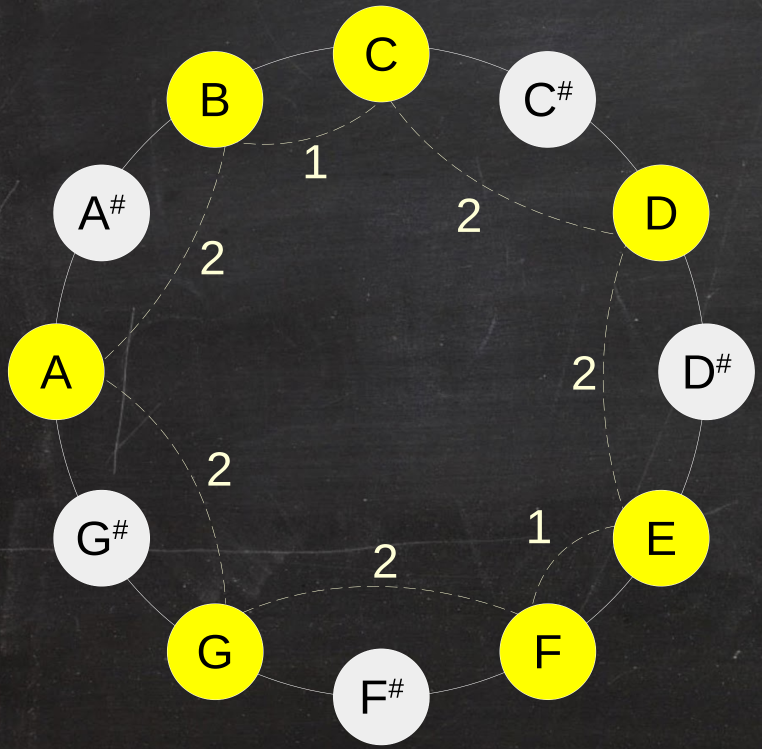
example



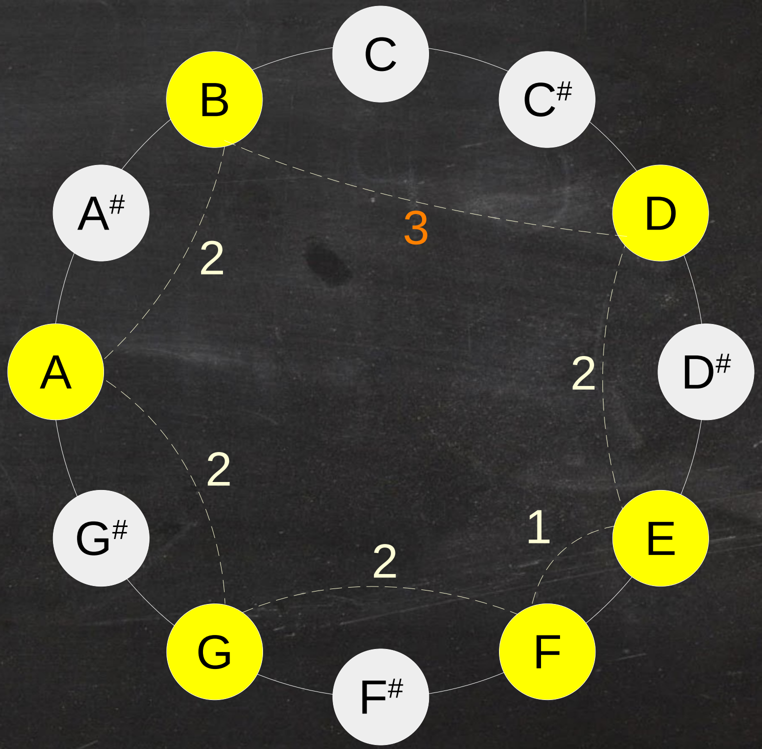
counterexample

maximal non-chromatic scales

A non-chromatic scale is **maximal** if it is not contained in any other non-chromatic scale.



example

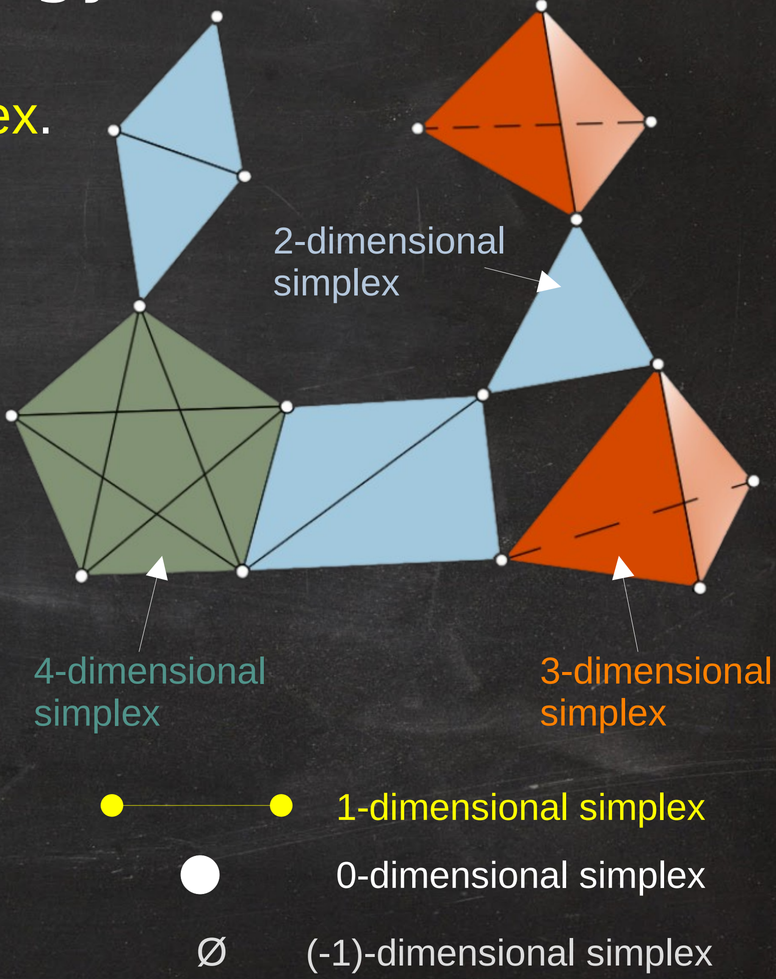


counterexample

Which scales are maximal non-chromatic scales?

algebraic topology

The non-chromatic scales form a simplicial complex.



A **simplicial complex** on a ground set *G* is a set *K* of consisting of finite subsets of *G* such that:
for every set *M* in *K* and every subset *T* of *M*,
we have that *T* is also in *K*.

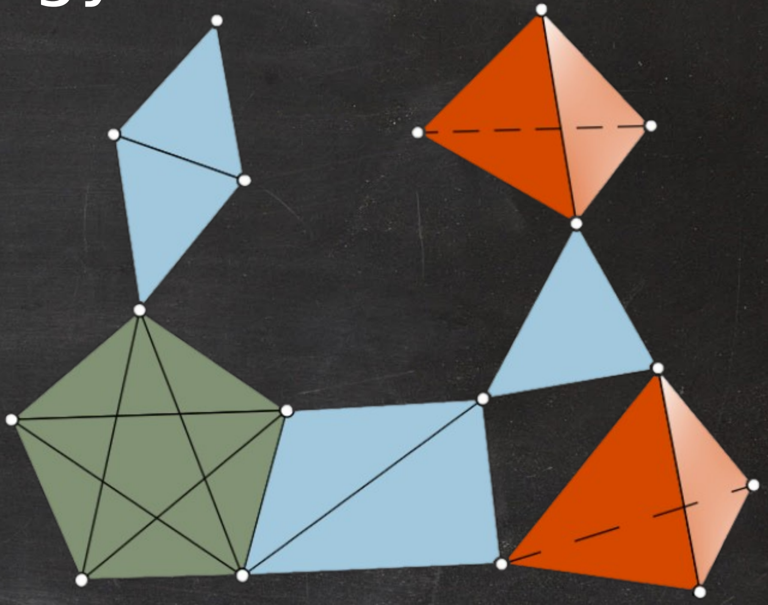
Example:

- *G* = { 0, 1, 2 }
- *K1* = { {0,1}, {2} } is **not** a simplicial complex
- *K2* = { {0,1}, {0}, {1}, {2}, Ø } is a simplicial complex

algebraic topology

The **f-vector** of a simplicial complex K is the list $(f_{-1}, f_0, f_1, f_2, \ldots)$ where f_n denotes the number of n-dimensional simplices in K.

f_{-1}	f_0	f_1	f_2	f_3	f_4
1	17	34	23=5+8+10	7=2+5	1



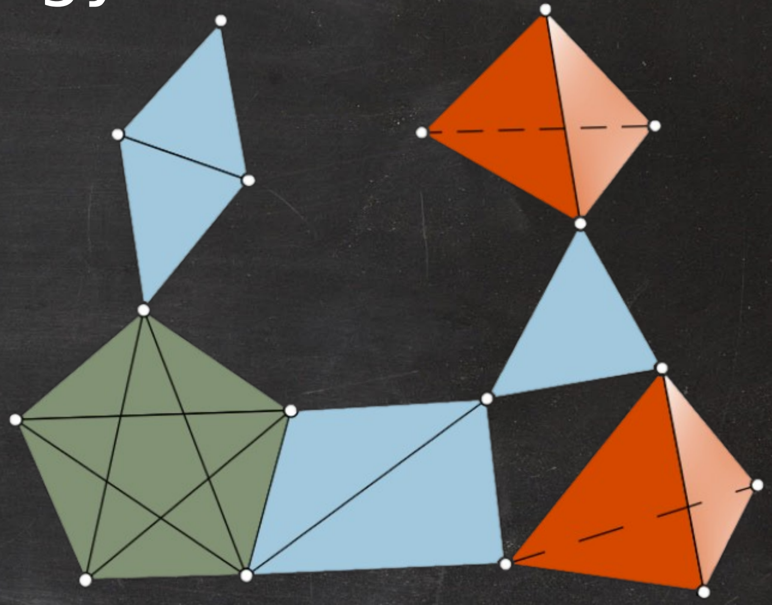
The non-chromatic scales form a simplicial complex.

f-vector:	f_{-1}	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7
#scales	1	12	66	208	399	456	282	72	3
#notes	0	1	2	3	4	5	6	7	8

Obs.:
There is no non-chromatic scale with 9 or more notes.

algebraic topology

A set *M* in a simplicial complex *K* is a **facet** of *K* if there is no other set in *K* that contains *M*.

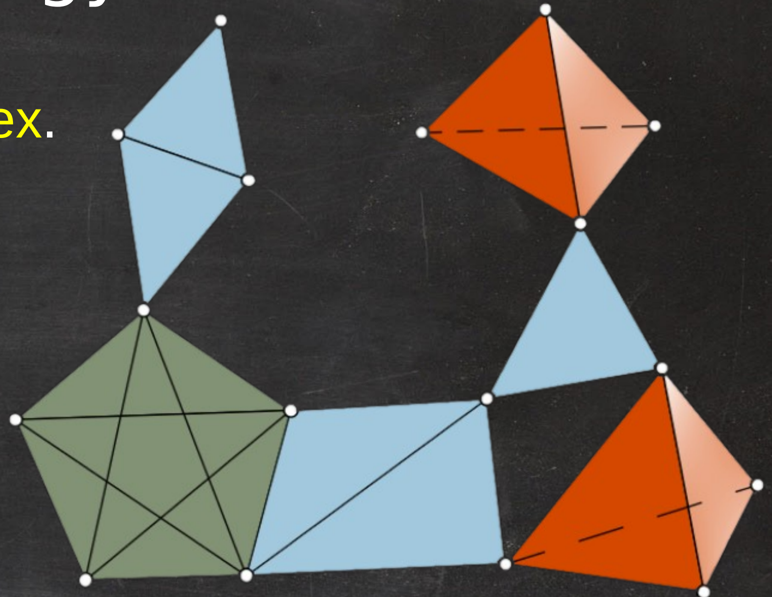


5 2-dimensional facets
2 3-dimensional facets
1 4-dimensional facet

The non-chromatic scales form a simplicial complex.
Its facets are the maximal non-chromatic scales!

algebraic topology

The non-chromatic scales form a simplicial complex.



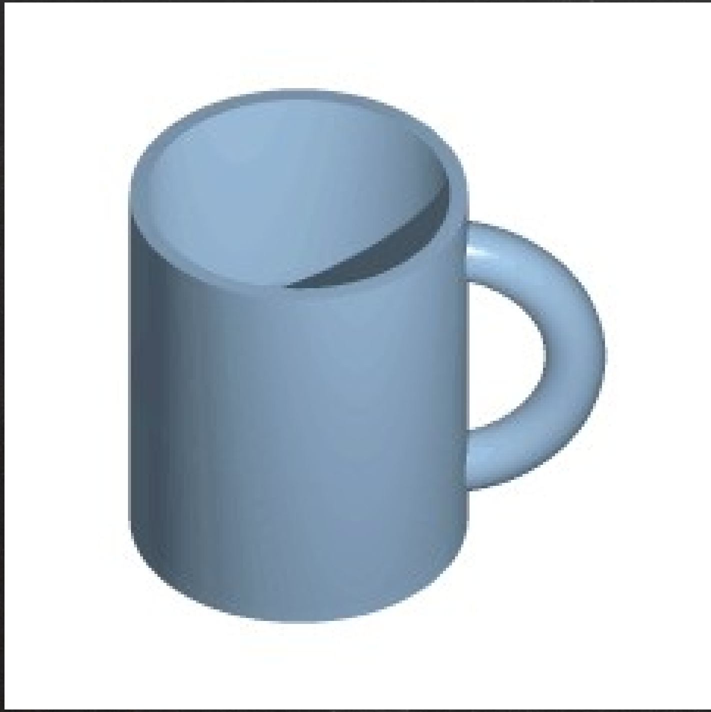
f-vector:

#scales	1	12	66	208	399	456	282	72	3
#notes	0	1	2	3	4	5	6	7	8

57 facets:

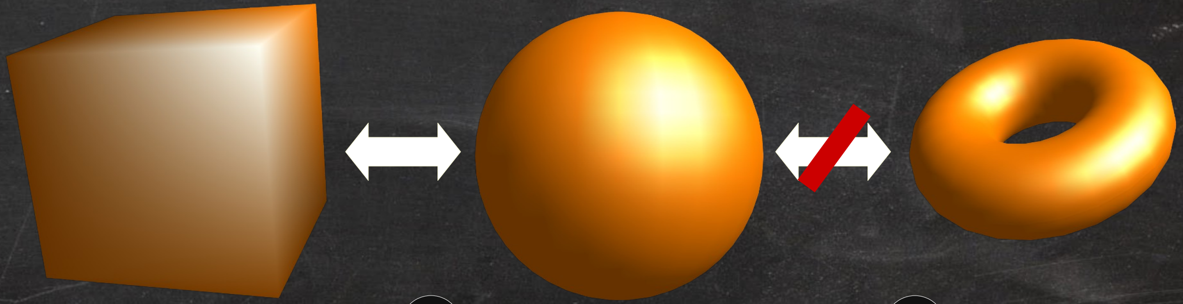
#notes	name	#scales
8	diminished	3
7	melodic (classical) major	12
7	melodic minor	12
7	harmonic minor	12
7	harmonic major	12
6	whole-tone	2
6	half-tone / augmented	4

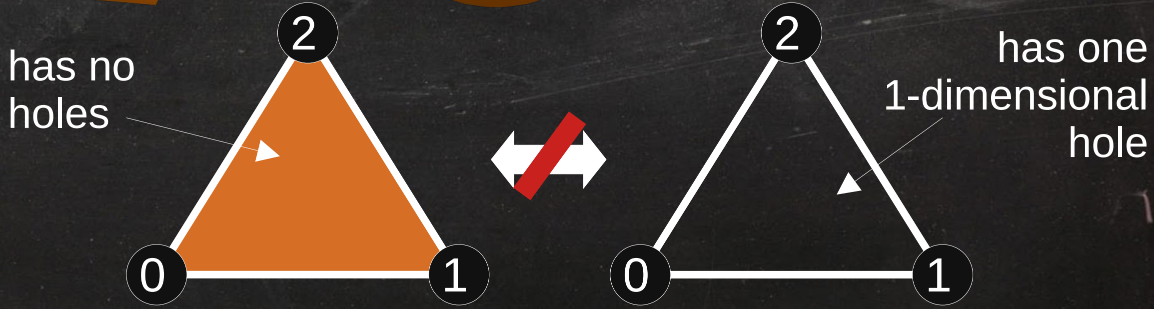
counting holes



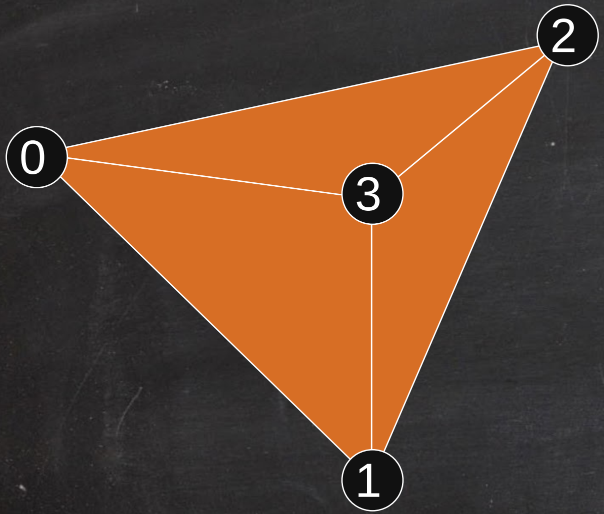
Which properties of spaces are preserved under continuous deformations?
(such as bending or stretching)

number of “holes” !



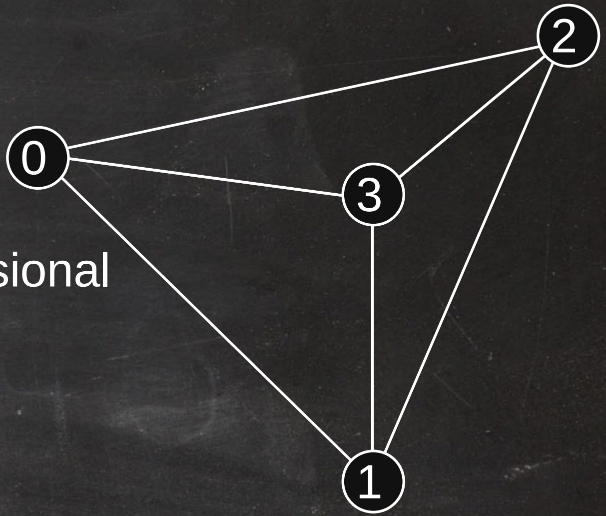


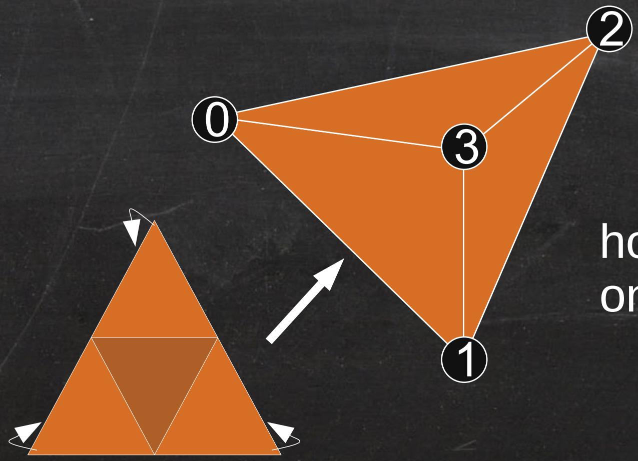
counting holes



solid object:
no holes

three
1-dimensional
holes





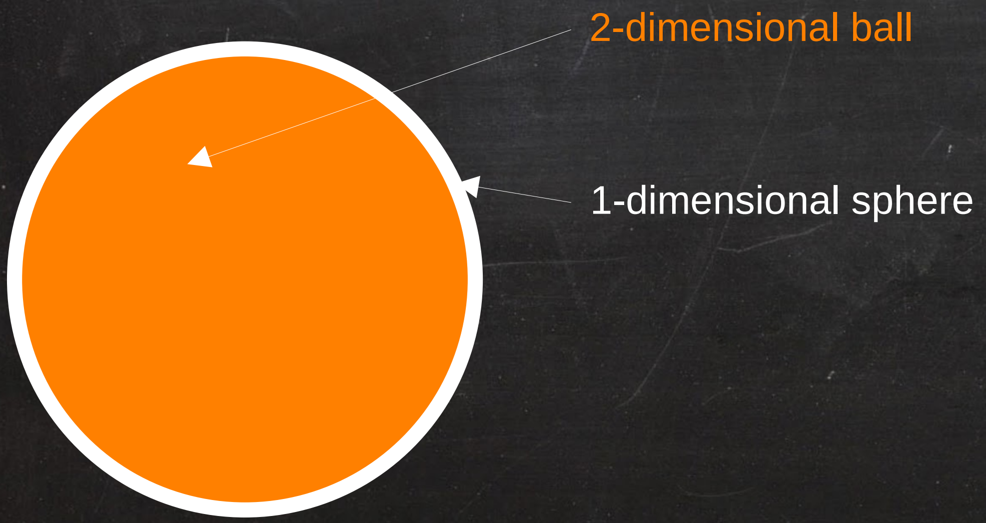
hollow object:
one 2-dimensional hole

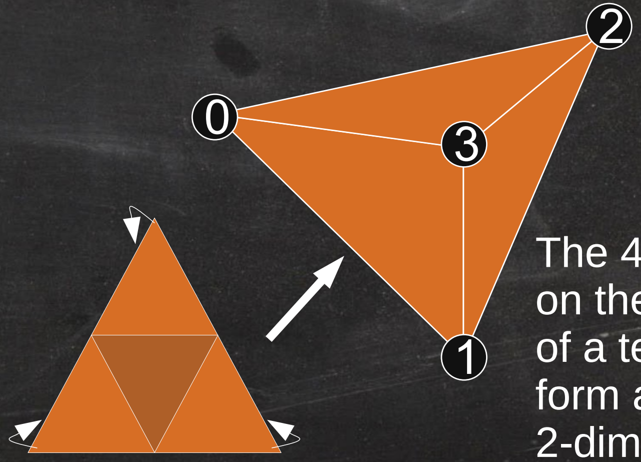
non-chromatic holes

The non-chromatic scales form a simplicial complex.
It has three 5-dimensional holes.

$H_5 = \mathbf{Q}^3$
$H_n = \{0\}$ for $n \neq 5$

- The boundary of each hole is formed by hexatonics (= scales with 6 notes).
- From the topological point of view, these hexatonics form a 5-dimensional sphere (= boundary of a 6-dimensional ball).

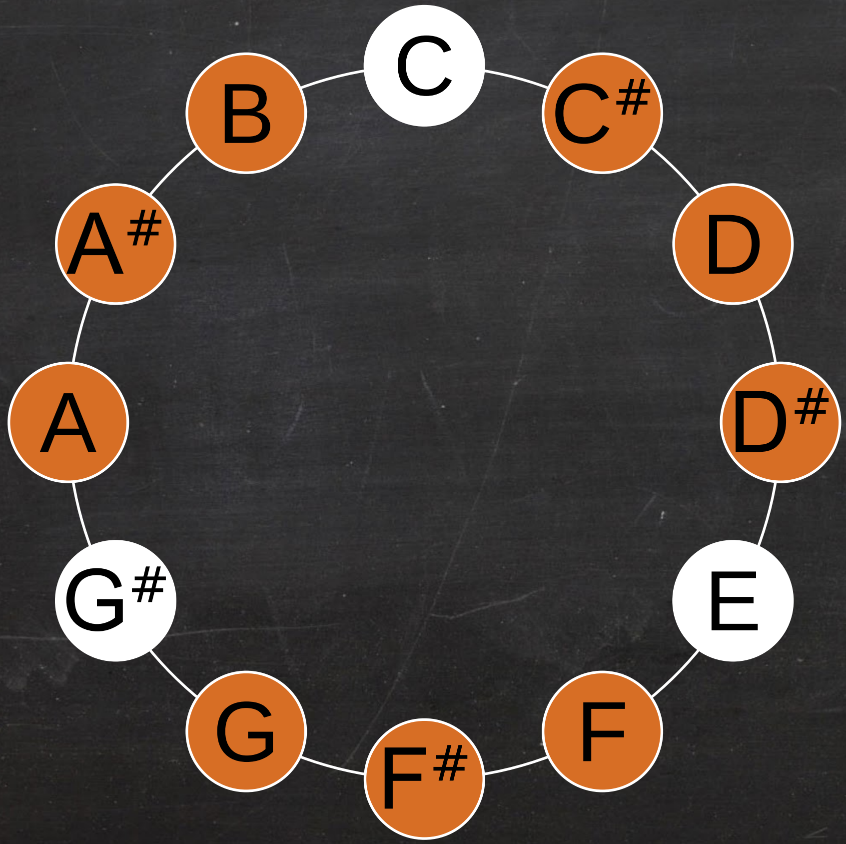




The 4 triangles on the boundary of a tetrahedron form a topological 2-dimensional sphere.

non-chromatic holes

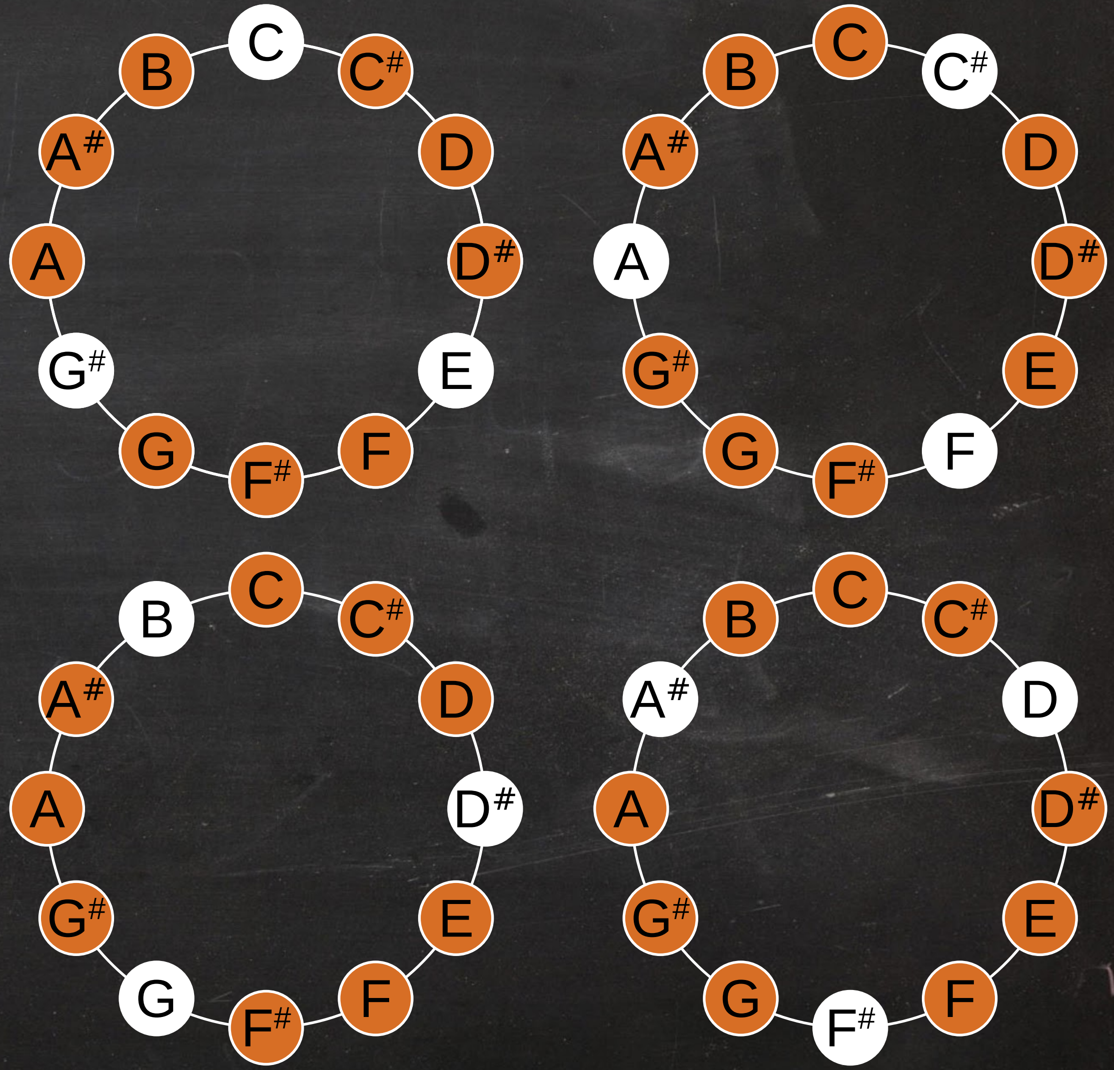
Messiaen’s 9-note scale contains 27 non-chromatic hexatonics.



These hexatonics form the boundary of one hole!

non-chromatic holes

There are 4 Messiaen scales:



But the simplicial complex of non-chromatic scales has only 3 holes!

Intuitive reason:
One can only “see” 3 holes simultaneously.

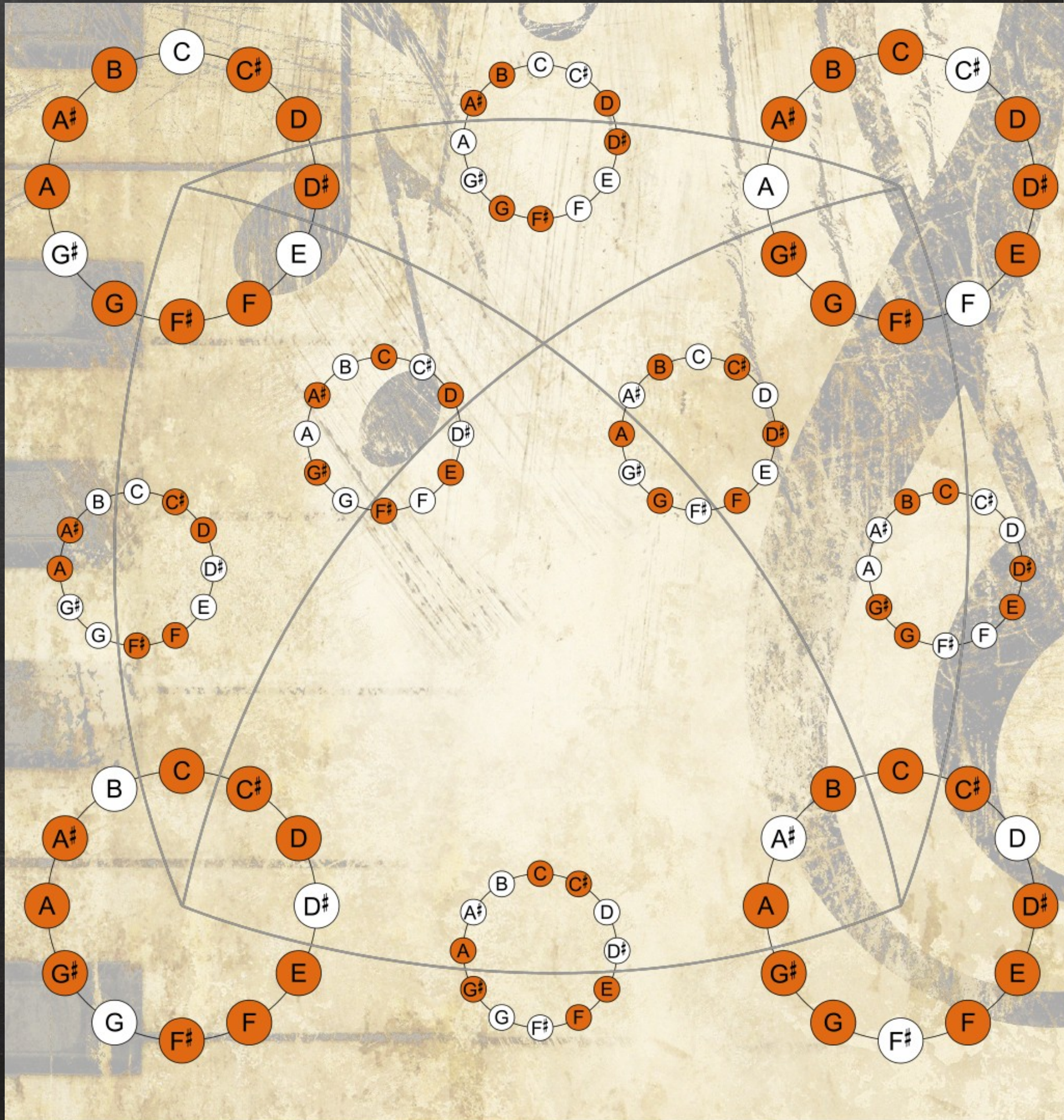
Mathematical reason:
The 4 Messiaen spheres are linearly dependent.
Any 3 of them are linearly independent.

arrangement of non-chromatic holes

The intersection of 2 Messiaen-spheres is a maximal non-chromatic scale:

57 facets:

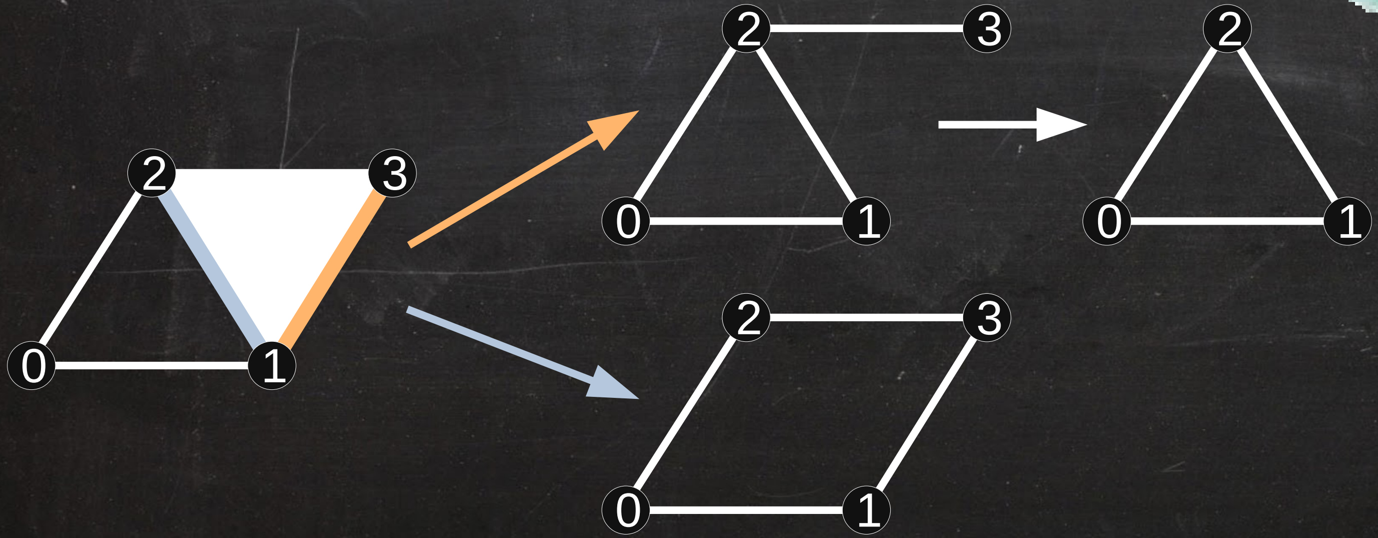
#notes	name	#scales
8	diminished	3
7	melodic (classical) major	12
7	melodic minor	12
7	harmonic minor	12
7	harmonic major	12
6	**whole-tone**	2
6	**half-tone / augmented**	4



where are the facets with 7 or 8 notes?

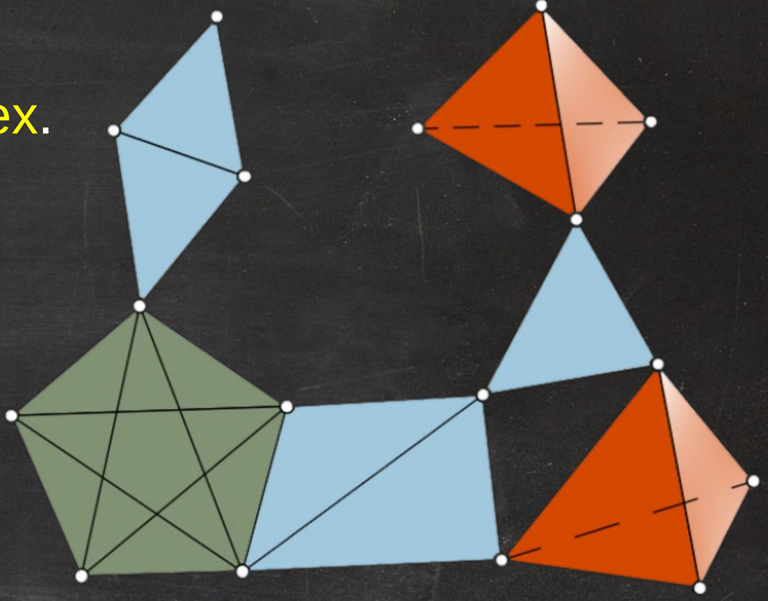
They sit on the “outside” of the holes!

We can remove them from the simplicial complex using **collapses** without changing the topology.



summary

The non-chromatic scales form a simplicial complex.



f-vector:

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57 facets:

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Simplicial homology:

Reduction via collapses
= getting rid of scales bigger than hexatonics without changing the topology

$H_5 = \mathbf{Q}^3$ ⟷ 3 "holes" with remaining hexatonic scales on boundary
$H_n = \{0\}$ for $n \neq 5$

↑

extensions of 10 basic pentatonic forms