Rational Polypols

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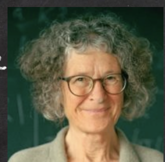

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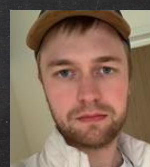

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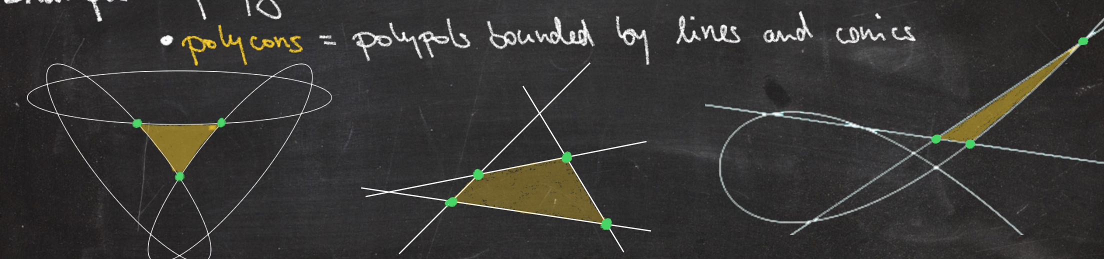
Polypols

Eugene Wachspress 1975: A polypol is a "closed planar figure bounded by algebraic curves."

Example: • polygons

• polycons = polypols bounded by lines and conics



Wachspress' goal: define barycentric coordinates on polypols

⟿ tool: adjoint curve of a polypol

Outline

I adjoint curves of rational polypols ⟷ canonical forms of positive geometries III

barycentric coordinates

scattering amplitudes (quantum physics)

Wachspress' conjecture: adjoint is outside of polypol

II real topology of adjoints

hyperbolicity

Rational Polypols

Def: A polypol P is given by $k \geq 2$ irreducible curves $C_1, C_2, \ldots, C_k \subseteq \mathbb{P}^2_{\mathbb{C}}$ and points $v_{12} \in C_1 \cap C_2$, $v_{23} \in C_2 \cap C_3$, …, $v_{k1} \in C_k \cap C_1$ such that

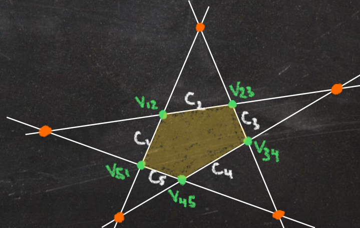
a) v_{ij} is smooth on C_i and C_j, and

b) C_i and C_j intersect transversally at v_{ij}.

- $V(P) := \{v_{12}, v_{23}, \ldots, v_{k1}\}$ are the vertices of P.
- Let $C := C_1 \cup C_2 \cup \ldots \cup C_k$. $R(P) := \mathrm{Sing}(C) \setminus V(P)$ is the set of residual points of P.

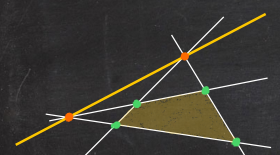
Def: P is rational if $C_1, C_2, \ldots, C_k$ are rational.

Example: polygons

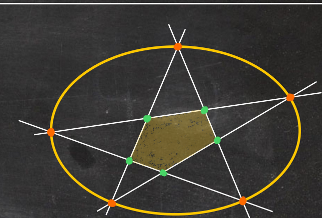


Wachspress' adjoints

Def: The adjoint curve $A_P \subseteq \mathbb{P}^2_{\mathbb{C}}$ of a polygon P in $\mathbb{P}^2_{\mathbb{C}}$ is the unique curve of minimal degree passing through $R(P)$.



$\deg A_P = |V(P)| - 3$



Thm: A polypol P given by rational nodal curves $C_1, C_2, \dots, C_k \subseteq \mathbb{P}^2_{\mathbb{C}}$ that intersect transversally has a unique curve $A_P \subseteq \mathbb{P}^2_{\mathbb{C}}$ of degree $\sum_{i=1}^{k} \deg C_i - 3$ passing through $R(P)$.

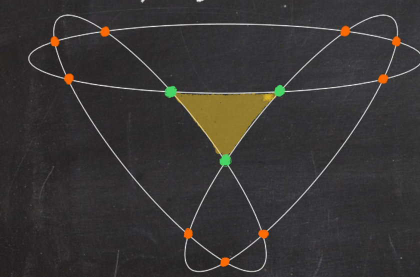
adjoint curve of P

Wachspress' adjoints

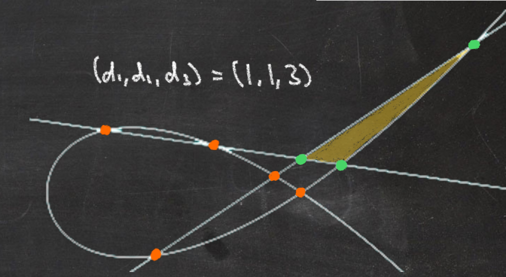
$d_i := \deg C_i$

$d := d_1 + d_2 + \dots + d_k$

$(d_1, d_2, d_3) = (2,2,2)$



$(d_1, d_1, d_3) = (1,1,3)$



Wachspress' adjoints

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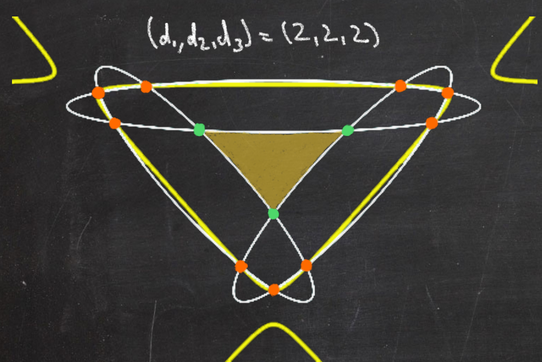

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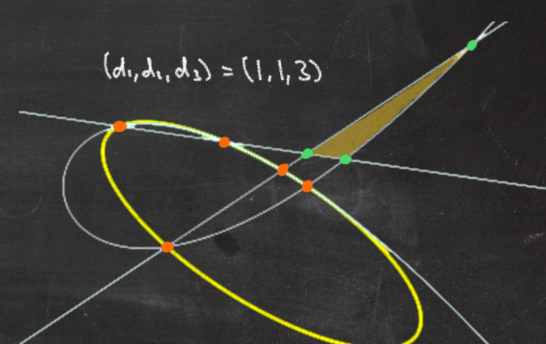
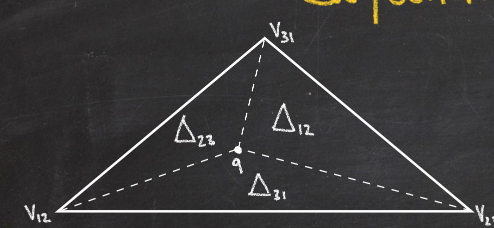


There is a unique adjoint curve A_P of degree $d-3$ for every rational polypol P, without restricting to only nodal singularities and transversal intersections, by requiring appropriate multiplicities of A_P at the residual points.

Barycentric Coordinates



$$\beta_{V_{ij}}(q) = \frac{\mathrm{area}(\Delta_{ij})}{\mathrm{area}(\Delta_{12}) + \mathrm{area}(\Delta_{23}) + \mathrm{area}(\Delta_{23})}$$

Def: Let $P \subseteq \mathbb{R}^2$ be a convex polygon. A set of function $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called generalized barycentric coordinates for P if, for all $q \in P^\circ$,

a) $\forall u \in V(P): \beta_u(q) > 0$

b) $\sum_{u \in V(P)} \beta_u(q) = 1$, and

c) $\sum_{u \in V(P)} \beta_u(q) u = q$.

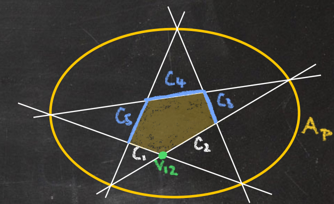
Barycentric coordinates for triangles are uniquely determined by a) – c).

This is not true for other polygons!

Barycentric Coordinates

Let $P \subseteq \mathbb{R}^2$ be a convex polygon defined by lines $C_1, \dots, C_k$ and vertices $v_{12}, v_{23}, \dots, v_{k1}$.

- Fix $l_i \in \mathbb{R}[x,y]$ such that $C_i = \mathcal{Z}(l_i)$.
- Fix $\alpha_P \in \mathbb{R}[x,y]$ such that $A_P = \mathcal{Z}(\alpha_P)$.



$$\beta_{v_{12}} = \frac{l_3 l_4 l_5}{\alpha_P}$$

Def: The Wachspress coordinates of P are

$$\beta_{v_{ij}}(q) = n_{ij} \frac{\prod_{m=1, m \neq i,j}^{k} l_m(q)}{\alpha_P(q)}.$$

n_{ij}: normalization constant such that $\beta_{v_{ij}}(v_{ij}) = 1$.

Wachspress provides similar construction, with adjoint as denominator, for regular rational polypols.

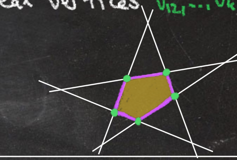
Wachspress' Conjecture

known for polygons

Conj: The adjoint curve A_P of a regular rational polypol $P \subseteq \mathbb{R}^2$ does not intersect its interior.

Let P be a rational polypol defined by real curves $C_1, \dots, C_k$ and real vertices $v_{12}, \dots, v_{k1}$.

- The i-th side of P is the real segment of C_i from $v_{i-1,i}$ to $v_{i,i+1}$.
- The union of the sides bounds a simply connected region $P_{\geq 0}$.

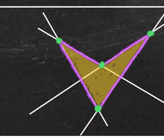


Def: P is regular if

a) all points on its sides, except its vertices, are smooth on $C = C_1 \cup \dots \cup C_k$, and

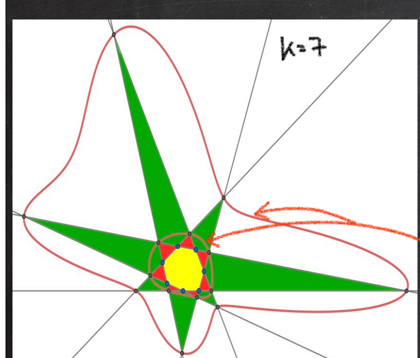
b) C does not pass through the interior of $P_{\geq 0}$.

Example: A polygon is regular iff it is convex.



Hyperbolic Adjoints

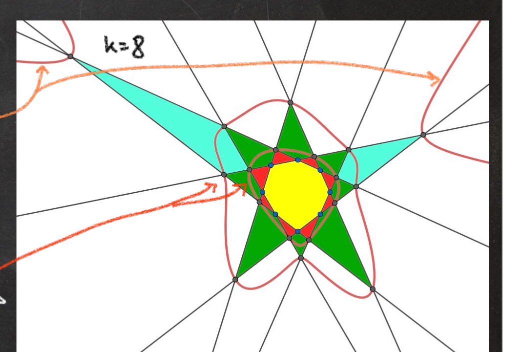
Thm: The adjoint curve of a convex polygon is hyperbolic.



$k = |V(P)|$

1 pseudoline if k is even

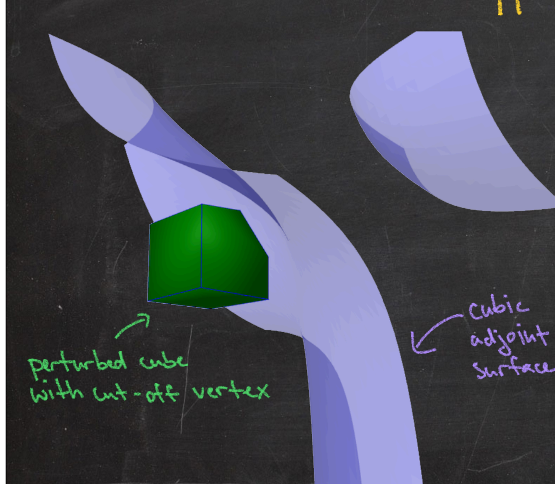
$\left\lfloor \frac{k-3}{2} \right\rfloor$ nested ovals

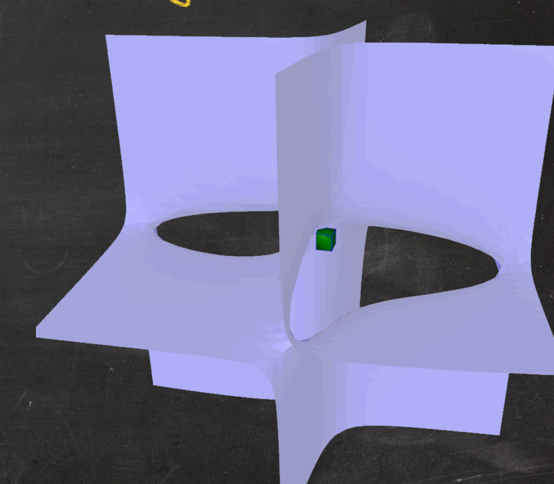


The i-th oval passes through the intersection points of pairs of edges of distance $i+1$.

The pseudoline ———— " ———— opposite edges.

Non-Hyperbolic Adjoints





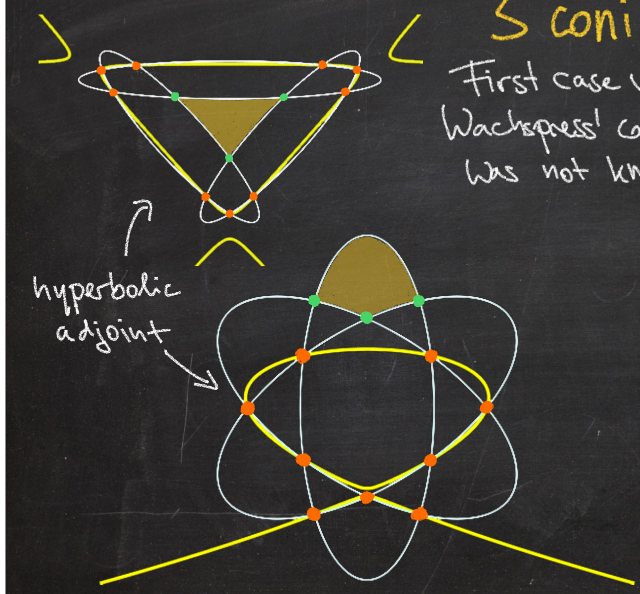
perturbed cube with cut-off vertex

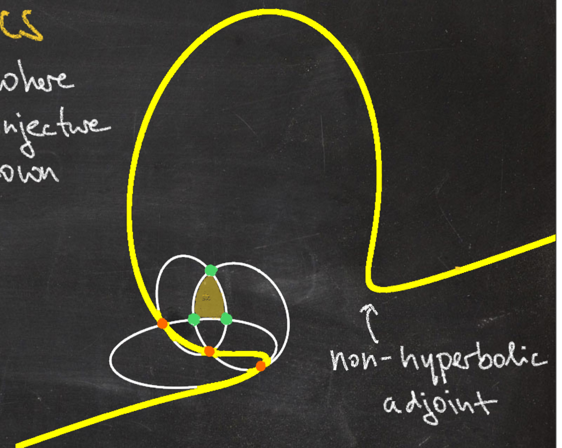
cubic adjoint surface

For convex 3D polytopes, the adjoint surface is typically not hyperbolic.

3 conics

First case where Wachspress' conjecture was not known





Thm: The adjoint curve of a regular polypol defined by 3 ellipses does not intersect its interior.

Outline

I ✓ adjoint curves of rational polypols ⟷ canonical forms of positive geometries III

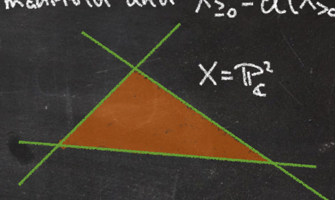
- barycentric coordinates
- Wachspress' conjecture: adjoint is outside of polypol
- scattering amplitudes (quantum physics)

II ✓ real topology of adjoints

- hyperbolicity

Positive Geometries (Arkani-Hamed, Bai, Lam 2017)

Let:
- X be a projective, complex, irreducible, n-dimensional variety,
- $X_{\geq 0} \subseteq X(\mathbb{R})$ be a closed semi-algebraic subset such that
- $X_{>0} = \mathrm{Int}(X_{\geq 0})$ is an open oriented n-dimensional manifold and $X_{\geq 0} = \mathrm{cl}(X_{>0})$.
- $\partial X_{\geq 0} = X_{\geq 0} \setminus X_{>0}$
- ∂X = Zariski closure of $\partial X_{\geq 0}$ in X
  $= C_1 \cup C_2 \cup \ldots \cup C_k$ irreducible components
- $C_{i,\geq 0} = \mathrm{cl}(\mathrm{Int}(C_i \cap X_{\geq 0}))$



Def: $(X, X_{\geq 0})$ is a positive geometry if there is a unique non-zero rational n-form $\Omega(X, X_{\geq 0})$, called its canonical form, satisfying:
- If $n=0$, $X = X_{\geq 0}$ = point and $\Omega(X, X_{\geq 0}) = \pm 1$.
- If $n>0$, $(C_i, C_{i,\geq 0})$ is a positive geometry s.t. $\mathrm{Res}_{C_i} \Omega(X, X_{\geq 0}) = \Omega(C_i, C_{i,\geq 0})$ $\forall i$, and $\Omega(X, X_{\geq 0})$ is holomorphic on $X \setminus (C_1 \cup \ldots \cup C_k)$.

Positive Geometries

Example:
- $n=1 \Rightarrow X$ rational curve
  $\Rightarrow X_{\geq 0}$ = union of closed intervals
  $\Omega(\mathbb{P}^1_{\mathbb{C}}, [a,b]) = \frac{b-a}{(b-x)(x-a)} dx$

- $(Gr(k,n), Gr(k,n)_{\geq 0})$ totally nonnegative Grassmannian

Conjecture: Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^{k+m}$ be a linear map, $m \geq 0$, $k+m \leq n$.
$\rightsquigarrow \tilde{\varphi} : Gr(k,n) \dashrightarrow Gr(k, k+m)$.
The Grassmann polytope (Lam 2015) $(Gr(k,k+m), \tilde{\varphi}(Gr(k,n)_{\geq 0}))$ is a positive geometry.

Special case: If the matrix of φ has positive maximal minors, the Grassmann polytope is called the (tree) amplituhedron $\mathcal{A}_{n,k,m}(\varphi)$.

For $m=4$, it encodes the scattering amplitude of n interacting particles, $k+2$ have helicity −, the others helicity +. (Arkani-Hamed, Trnka 2013)

Rational Polypols are Positive Geometries

General problem for positive geometries: find formulae for $\Omega(X, X_{\geq 0})$

Now let $(X, X_{\geq 0})$ be a positive geometry where $X = \mathbb{P}^2_{\mathbb{C}}$.

$\Rightarrow C_1, \dots, C_k$ are rational curves

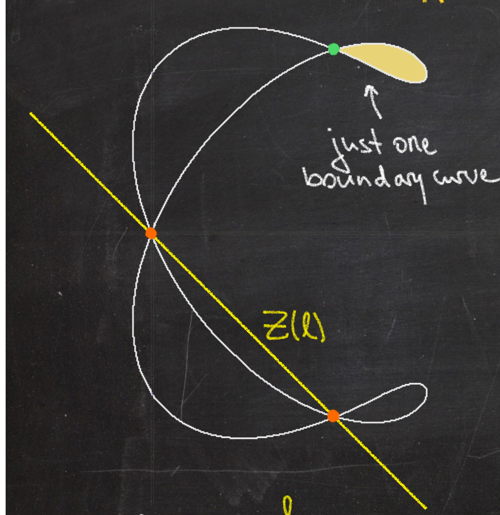
$\Rightarrow X_{\geq 0}$ is a generalized rational polypol.

Thm: Let $P_{\geq 0}$ be a real rational polypol with boundary curves $C_1, \dots, C_k$. Then $(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0})$ is a positive geometry with canonical form

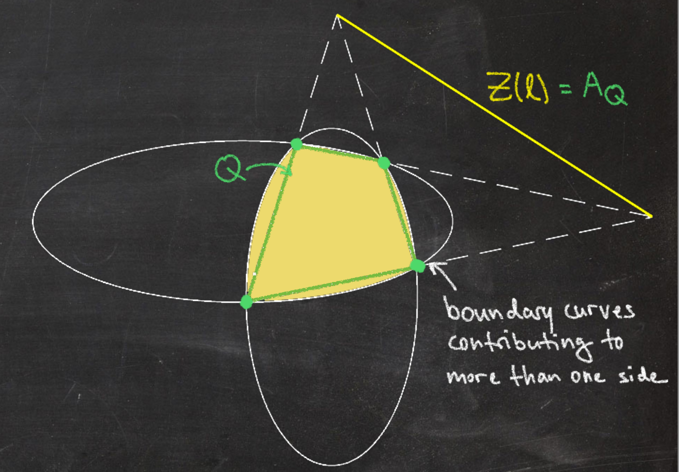
$$\Omega(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0}) = \eta \frac{\alpha_P}{f_1 f_2 \cdots f_k} dx \wedge dy$$

where $\alpha_P, f_1, \dots, f_k \in \mathbb{R}[x,y]$ such that $Z(\alpha_P) = A_P$, $Z(f_i) = C_i$, and η is a normalizing constant.

Non-Polypol Positive Geometries

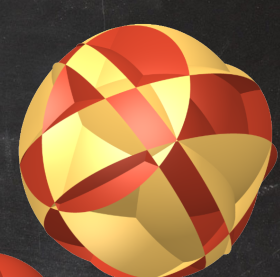


$$\Omega(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0}) = \frac{\ell}{f_1} dx \wedge dy$$



$$\Omega(\mathbb{P}^2_{\mathbb{C}}, P_{\geq 0}) = \frac{\ell}{f_1 f_2} dx \wedge dy$$

Thanks for your attention!



3D polypols