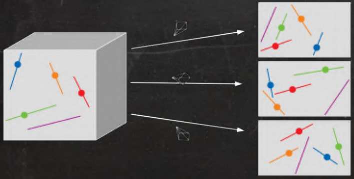
finding good minimal / optimization problems for SfM & triangulation

Kathlén Kohn

KTH VETENSKAP OCH KONST

WASP WALLENBERG AI, AUTONOMOUS SYSTEMS AND SOFTWARE PROGRAM



SF SG SWEDISH FOUNDATIONS' STARTING GRANT

Ragnar Söderbergs STIFTELSE

Göran Gustafssons Stiftelser

A Framework for Reducing the Complexity of Algebraic Optimization Problems

joint with **Felix Rydell**, Georg Bökman, Fredrik Kahl

2-view triangulation

Given

- F = fundamental matrix of camera pair and
- $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \times \mathbb{R}^2$ noisy image points,

we aim to solve

$$\min_{x_1, x_2, y_1, y_2} (x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 + (y_1 - \tilde{y}_1)^2 + (y_2 - \tilde{y}_2)^2,$$

$$[x_1, x_2, 1]\, F\, [y_1, y_2, 1]^\top = 0$$

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This optimization problem has 6 complex critical points generically.

Weighted triangulation

Can we find $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$ such that

$$\min_{x_1, x_2, y_1, y_2} \lambda_1 (x_1 - \tilde{x}_1)^2 + \lambda_2 (x_2 - \tilde{x}_2)^2 + \lambda_3 (y_1 - \tilde{y}_1)^2 + \lambda_4 (y_2 - \tilde{y}_2)^2,$$
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has less critical points?

Yes! We can bring it from 6 down to 2 :)

Step 1: Coordinate change

General idea: Diagonalize the constraint by translating and rotating.

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Concretely, find $t \in \mathbb{R}^4$ and $R \in \mathrm{SO}(4)$ such that in the new coordinates $z = R([x_1, x_2, y_1, y_2]^\top - t)$, the constraint $[x_1, x_2, 1]\, F\, [y_1, y_2, 1]^\top$ is of the form $q_1 z_1^2 + q_2 z_2^2 + q_3 z_3^2 + q_4 z_4^2$.

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General Proposition: Translation and rotation yield an equivalent Euclidean-distance minimization problem, with the same optimal value and the same number of critical points.

$$\min_{z_1,\ldots,z_4} (z_1 - \tilde{z}_1)^2 + (z_2 - \tilde{z}_2)^2 + (z_3 - \tilde{z}_3)^2 + (z_4 - \tilde{z}_4)^2,$$

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Exercise: In our case, $(q_1, q_2, q_3, q_4) = (a_1, -a_1, a_2, -a_2)$, where $a_1, a_2 \geq 0$ are the singular values of the top-left 2×2 block in F.

Step 2: Align weights with diagonal constraint

Theorem The number of critical points of

$$\min_{z_1,\ldots,z_4} \lambda_1(z_1-\tilde{z}_1)^2+\lambda_2(z_2-\tilde{z}_2)^2+\lambda_3(z_3-\tilde{z}_3)^2+\lambda_4(z_4-\tilde{z}_4)^2,$$

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is generically

- 2 if $\boldsymbol{\lambda}=(\mu a_1,\nu a_1,\mu a_2,\nu a_2)$ for some $\mu,\nu>0$,
- 4 if $(\lambda_1,\lambda_3)=\mu(a_1,a_2)$ for $\mu>0$ or $(\lambda_2,\lambda_4)=\nu(a_1,a_2)$ for $\nu>0$,
- 6 otherwise.

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We can solve weighted 2-view triangulation in closed form!

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What is the optimal choice of μ and ν?

Step 3: Least deviation from unweighted problem

Let $\boldsymbol{\lambda} = (a_1, \nu a_1, a_1, \nu a_2)$.
For every $\nu > 0$, let $\boldsymbol{z}(\nu)$ be the global minimum of the weighted problem.

Which $\boldsymbol{z}(\nu)$ minimizes the original objective $\sum(z_i(\nu) - \tilde{z}_i)^2$?

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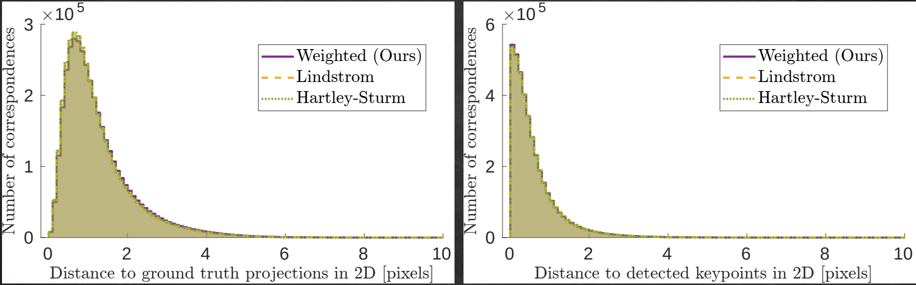
Which $\boldsymbol{z}(\nu)$ minimizes the original objective $\sum (z_i(\nu) - \tilde{z}_i)^2$?

Theorem: There is a unique such ν:

$$\nu = \frac{(\tilde{z}_2^2 + \tilde{z}_4^2)(a_1 \tilde{z}_1^2 + a_2 \tilde{z}_3^2)}{(\tilde{z}_1^2 + \tilde{z}_3^2)(a_1 \tilde{z}_2^2 + a_2 \tilde{z}_4^2)}$$

experiments



2D errors over correspondences from randomly sampled image pairs from the Pantheon dataset

3-view triangulation?

The analogous problem for camera triples (P_1, P_2, P_3) given noisy image points $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ is

$$\min_{x_1,x_2,y_1,y_2,z_1,z_2} (x_1 - \tilde{x}_1)^2 + (x_2 - \tilde{x}_2)^2 + (y_1 - \tilde{y}_1)^2 + (y_2 - \tilde{y}_2)^2 + (z_1 - \tilde{z}_1)^2 + (z_2 - \tilde{z}_2)^2,$$

$$(x_1, x_2, 1) \equiv P_1 X$$
$$(y_1, y_2, 1) \equiv P_2 X$$
$$(z_1, z_2, 1) \equiv P_3 X$$
for some $X \in \mathbb{P}^3$.

It has 47 critical points generically.

Can you find weights $\lambda_1, \ldots, \lambda_6 > 0$ such that the generic number of critical points is as low as possible? How low can it even get?

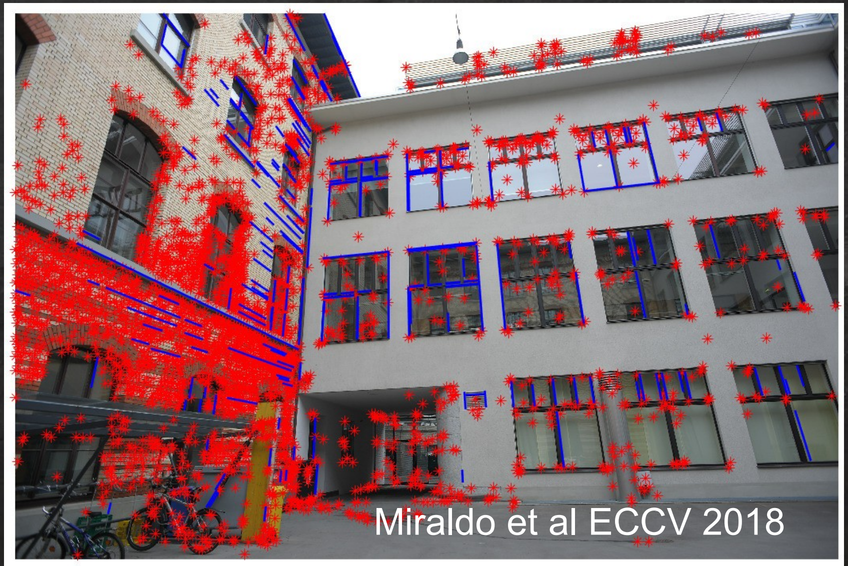
Point-Line Minimal Problems for SfM

joint with **Kim Kiehn**, Albin Ahlbäck

Fundamental Research Questions

1. Can we list **all** minimal problems?
2. How many solutions do they have?

We do not only want to work with points,
but also with lines and their incidences!



Our Result from 2019 with T. Duff, A. Leykin, T. Pajdla

We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image. for calibrated cameras.

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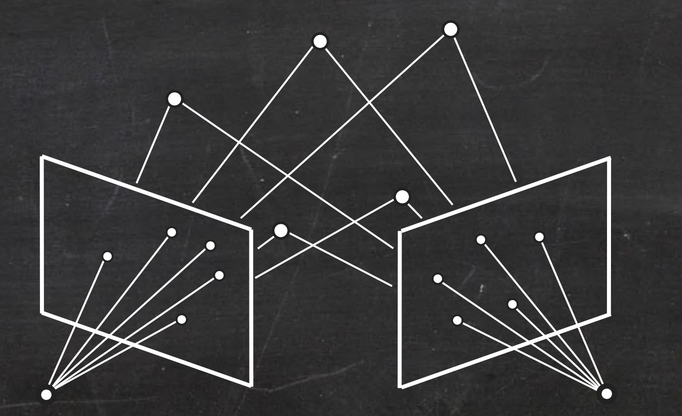
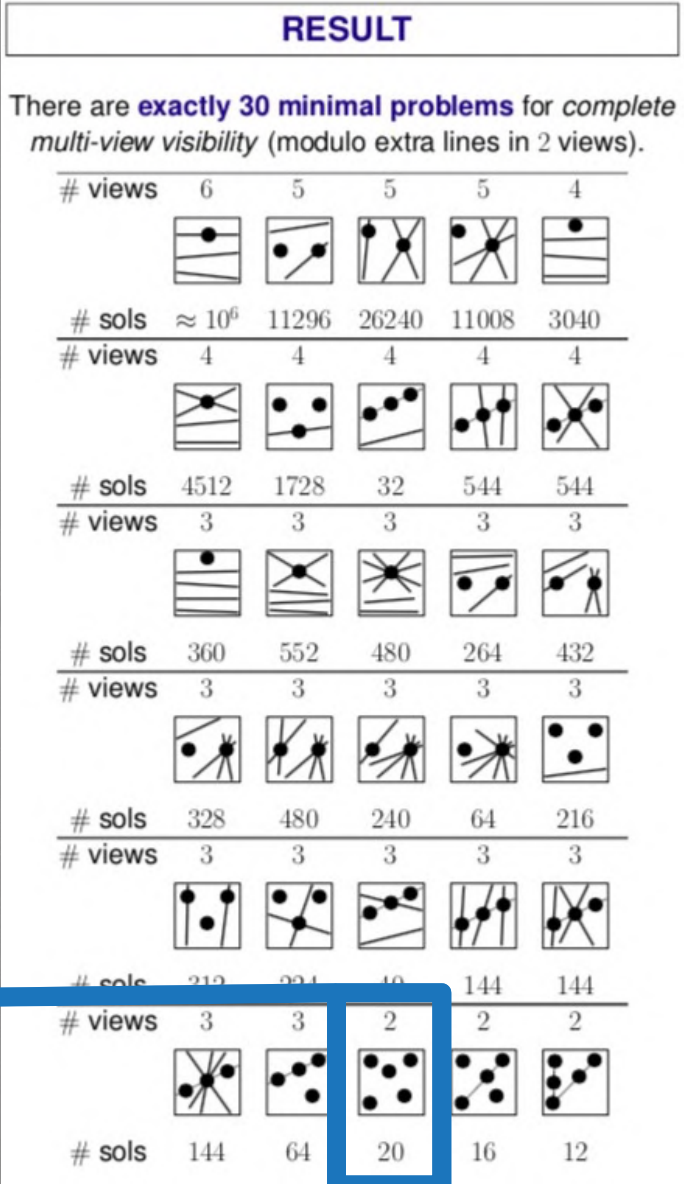


The 5-point problem has 20 solutions.



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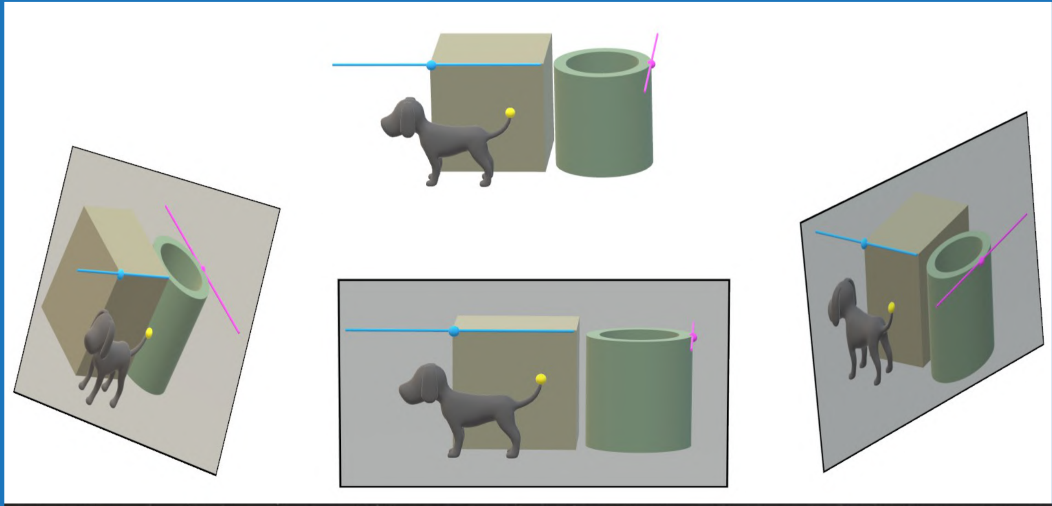
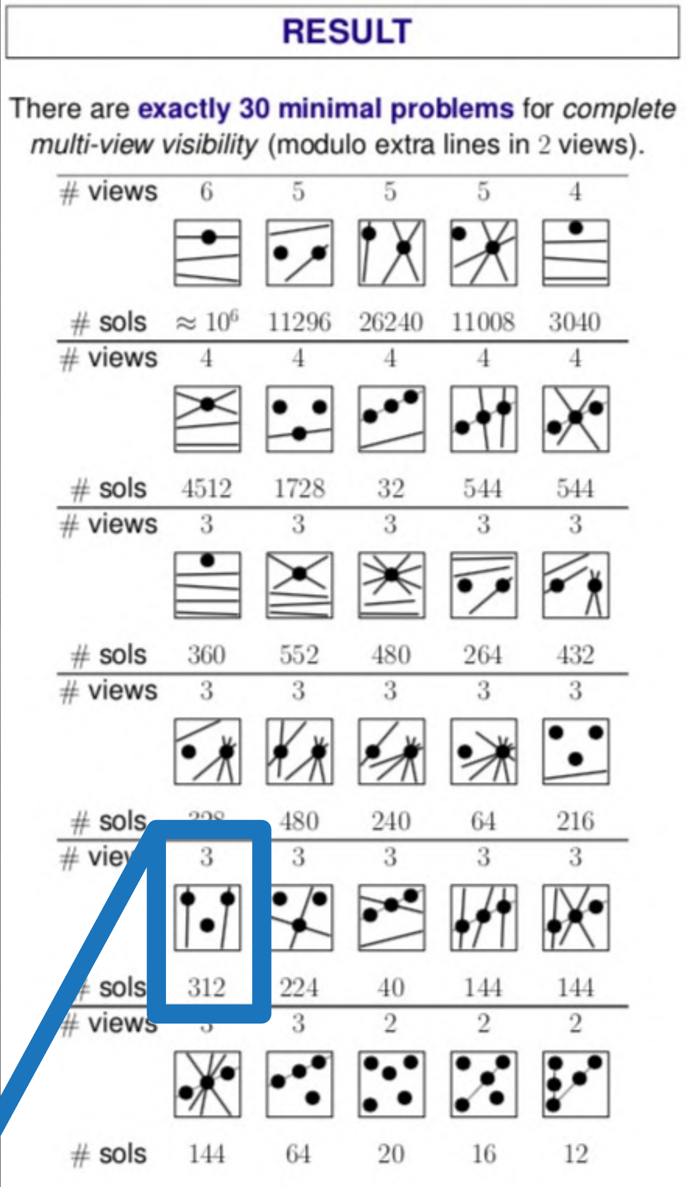
We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image.

First solver for such a high-degree problem based on state-of-the-art algorithms from **numerical algebraic geometry:**

TRPLP – Trifocal Relative Pose from Lines at Points, Fabbri et. al., CVPR 2020



This problem has 312 solutions
(counted over the complex numbers).

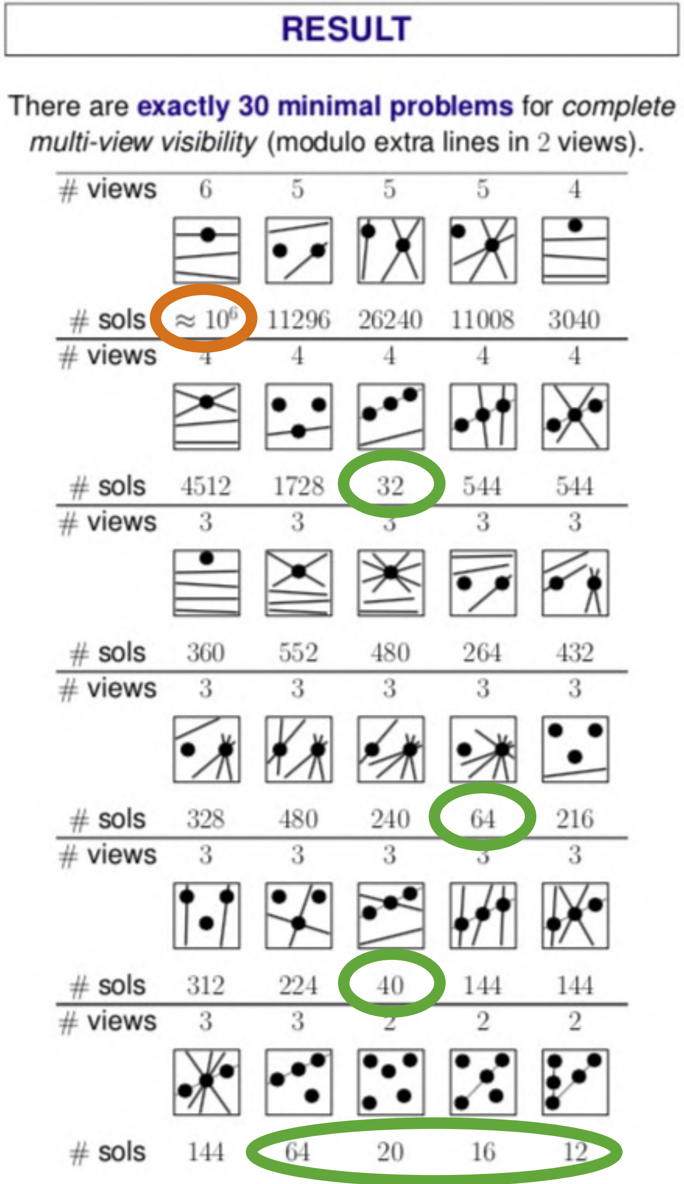


Our Result

We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image.

We **measure the complexity of each minimal problem** by computing its number of solutions

(counted over the complex numbers).



What about projective cameras?

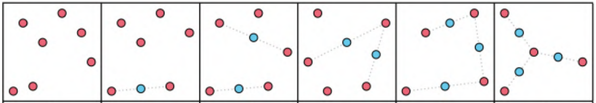
Theorem (K. Kiehn, A. Ahlbäck, K. Kohn): For projective cameras, all minimal problems involving points and lines are:

a) 2 cameras viewing one of the point-line arrangements in Table 1, plus arbitrarily many additional lines;

b) at least 2 cameras observing one of the 2 right-most point-line arrangements in Table 1;

c) one of the 285 PLPs in SM Section E (with 3–9 views).

Their degrees are given in Table 1 and SM Section E.



3	2	2	1	1	1

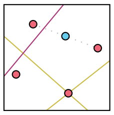
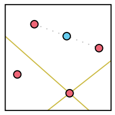
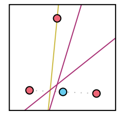
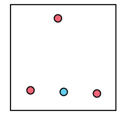
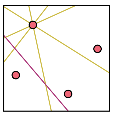
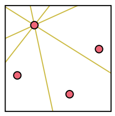
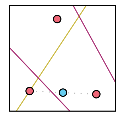
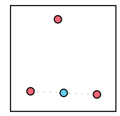
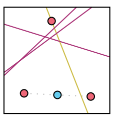
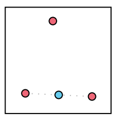
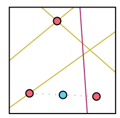
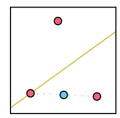
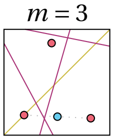
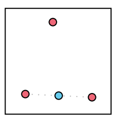
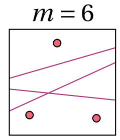
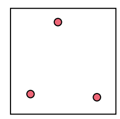
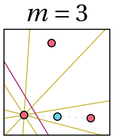
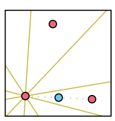
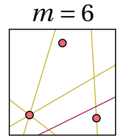
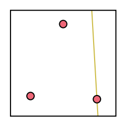
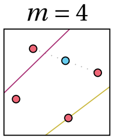
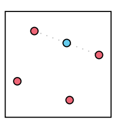
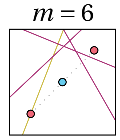
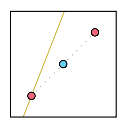
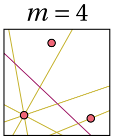
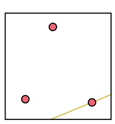
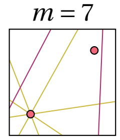
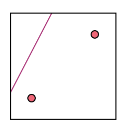
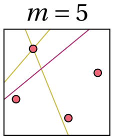
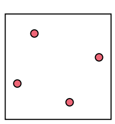
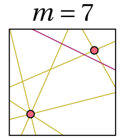
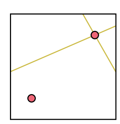
Problem	Subproblem	Constraints	Problem	Subproblem	Constraints
$m=3$  (4,1,1,2)	 (4,1,0,2)	1 point on free line with 0 in one coordinate	$m=5$  (3,1,2,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate
$m=3$  (4,0,1,4)	 (4,0,0,4)	1 point on free line with 0 in one coordinate	$m=5$  (3,1,2,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate
$m=3$  (3,1,3,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate	$m=5$  (3,1,1,3)	 (3,1,0,1)	1 point on each unused (free and adjacent) line with 0 in one coordinate
$m=3$  (3,1,3,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate	$m=6$  (3,0,3,0)	 (3,0,0,0)	1 point on each free line with 0 in one coordinate
$m=3$  (3,1,1,5)	 (3,1,0,5)	1 point on free line with 0 in one coordinate	$m=6$  (3,0,1,4)	 (3,0,0,1)	1 point on each unused (free and adjacent) line with 0 in one coordinate
$m=4$  (4,1,1,1)	 (4,1,0,0)	1 point on each unused (free and adjacent) line with 0 in one coordinate	$m=6$  (2,1,3,1)	 (2,1,0,1)	1 point on each free line with 0 in one coordinate
$m=4$  (3,0,1,5)	 (3,0,0,1)	1 point on each unused (free and adjacent) line with 0 in one coordinate	$m=7$  (2,0,2,4)	 (2,0,1,0)	1 point on each unused (free and adjacent) line with 0 in one coordinate
$m=5$  (4,0,1,2)	 (4,0,0,0)	1 point on each unused (free and adjacent) line with 0 in one coordinate	$m=7$  (2,0,1,6)	 (2,0,0,2)	1 point on each unused (free and adjacent) line with 0 in one coordinate

Table 6: List of non-minimal subproblems with overconstraint subsystems after elimination of the given subproblem. More details can be found in Example C.1 which is the first entry of this table.

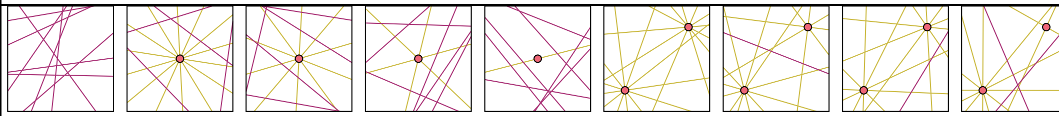
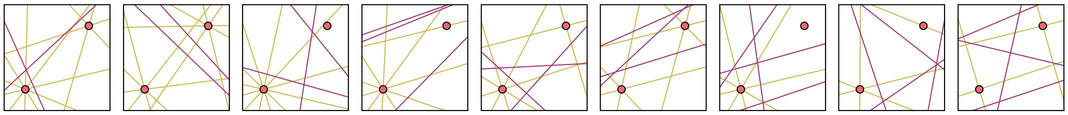
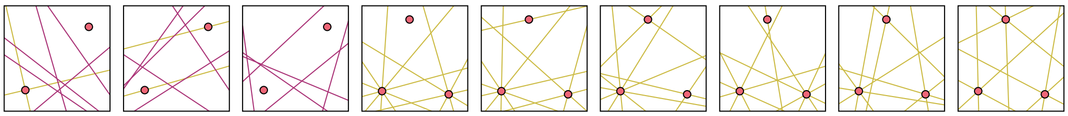
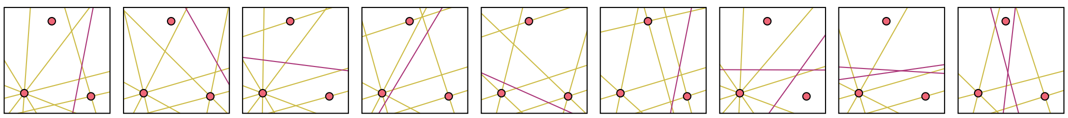
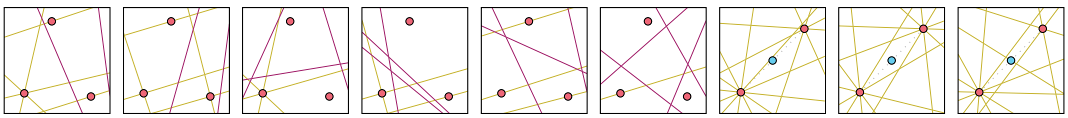
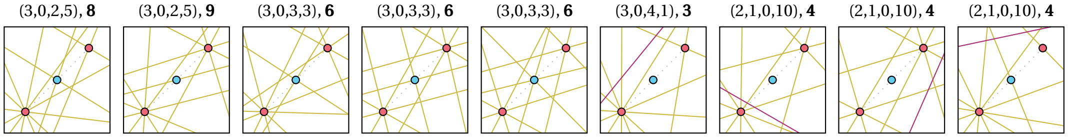
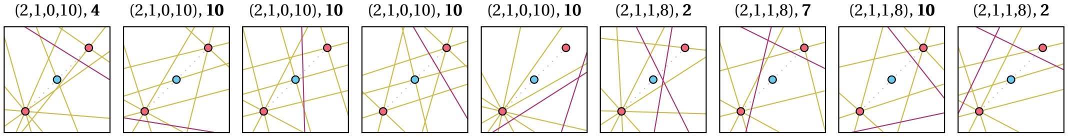
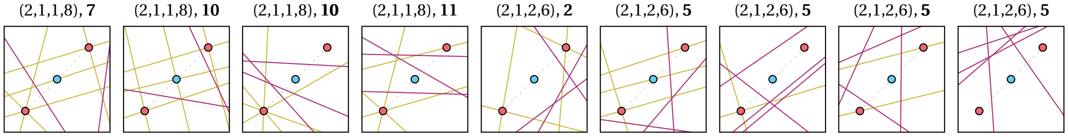
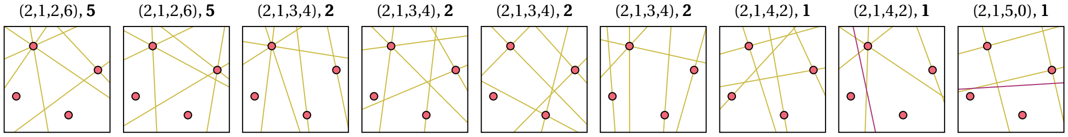
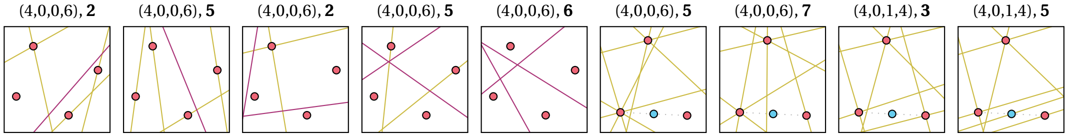
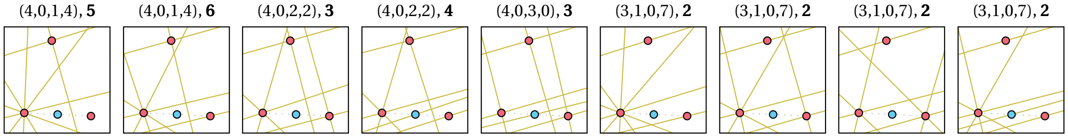
m	(p^f, p^d, l^f, l^a), algebraic degree								
3									
	(0,0,9,0), **36**	(1,0,4,7), **6**	(1,0,5,5), **23**	(1,0,6,3), **23**	(1,0,7,1), **15**	(2,0,0,12), **4**	(2,0,1,10), **6**	(2,0,1,10), **16**	(2,0,2,8), **4**
									
	(2,0,2,8), **12**	(2,0,2,8), **16**	(2,0,3,6), **2**	(2,0,3,6), **9**	(2,0,3,6), **15**	(2,0,3,6), **17**	(2,0,4,4), **9**	(2,0,4,4), **12**	(2,0,4,4), **13**
									
	(2,0,5,2), **8**	(2,0,5,2), **9**	(2,0,6,0), **7**	(3,0,0,9), **4**	(3,0,0,9), **4**	(3,0,0,9), **4**	(3,0,0,9), **10**	(3,0,0,9), **10**	(3,0,0,9), **12**
									
	(3,0,1,7), **2**	(3,0,1,7), **7**	(3,0,1,7), **2**	(3,0,1,7), **7**	(3,0,1,7), **10**	(3,0,1,7), **11**	(3,0,2,5), **2**	(3,0,2,5), **5**	(3,0,2,5), **7**
									
	(3,0,2,5), **8**	(3,0,2,5), **9**	(3,0,3,3), **6**	(3,0,3,3), **6**	(3,0,3,3), **6**	(3,0,4,1), **3**	(2,1,0,10), **4**	(2,1,0,10), **4**	(2,1,0,10), **4**
									
	(2,1,0,10), **4**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,1,8), **2**	(2,1,1,8), **7**	(2,1,1,8), **10**	(2,1,1,8), **2**
									
	(2,1,1,8), **7**	(2,1,1,8), **10**	(2,1,1,8), **10**	(2,1,1,8), **11**	(2,1,2,6), **2**	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,2,6), **5**
									
	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,4,2), **1**	(2,1,4,2), **1**	(2,1,5,0), **1**
									
	(4,0,0,6), **2**	(4,0,0,6), **5**	(4,0,0,6), **2**	(4,0,0,6), **5**	(4,0,0,6), **6**	(4,0,0,6), **5**	(4,0,0,6), **7**	(4,0,1,4), **3**	(4,0,1,4), **5**
									
	(4,0,1,4), **5**	(4,0,1,4), **6**	(4,0,2,2), **3**	(4,0,2,2), **4**	(4,0,3,0), **3**	(3,1,0,7), **2**	(3,1,0,7), **2**	(3,1,0,7), **2**	(3,1,0,7), **2**
									
	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **6**	(3,1,0,7), **5**

Table 7: Minimal problems with their associated degree.

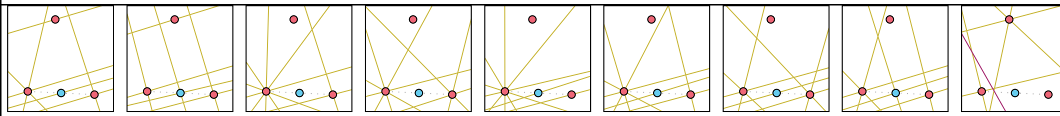
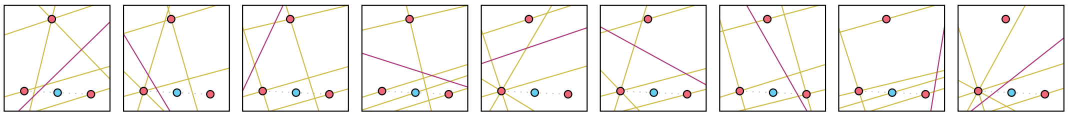
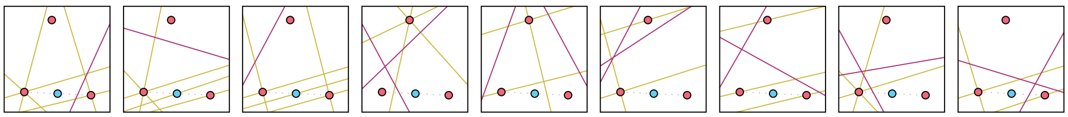
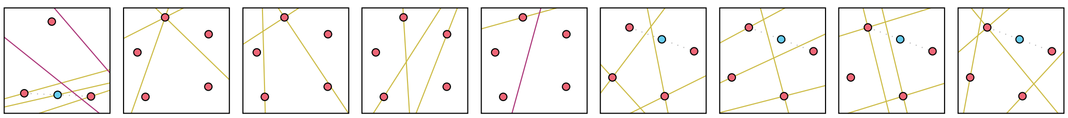
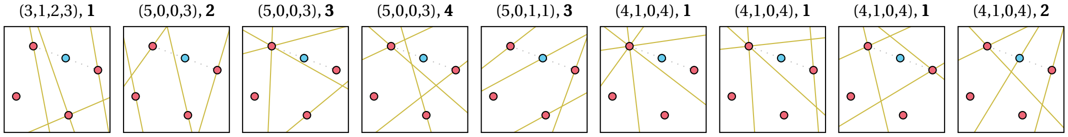
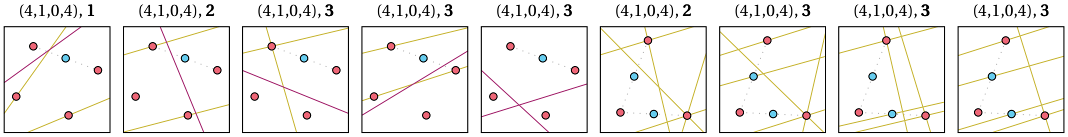
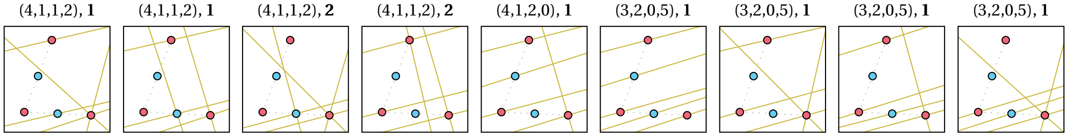
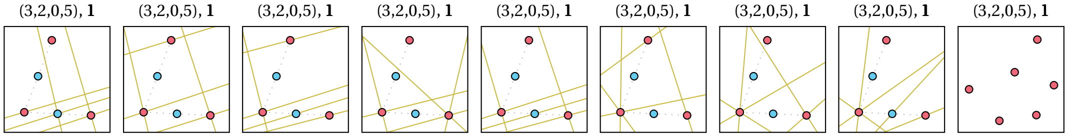
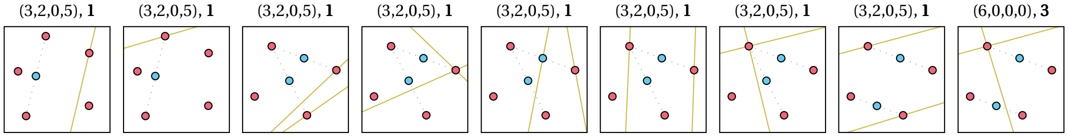
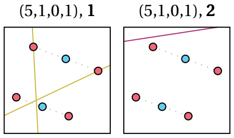
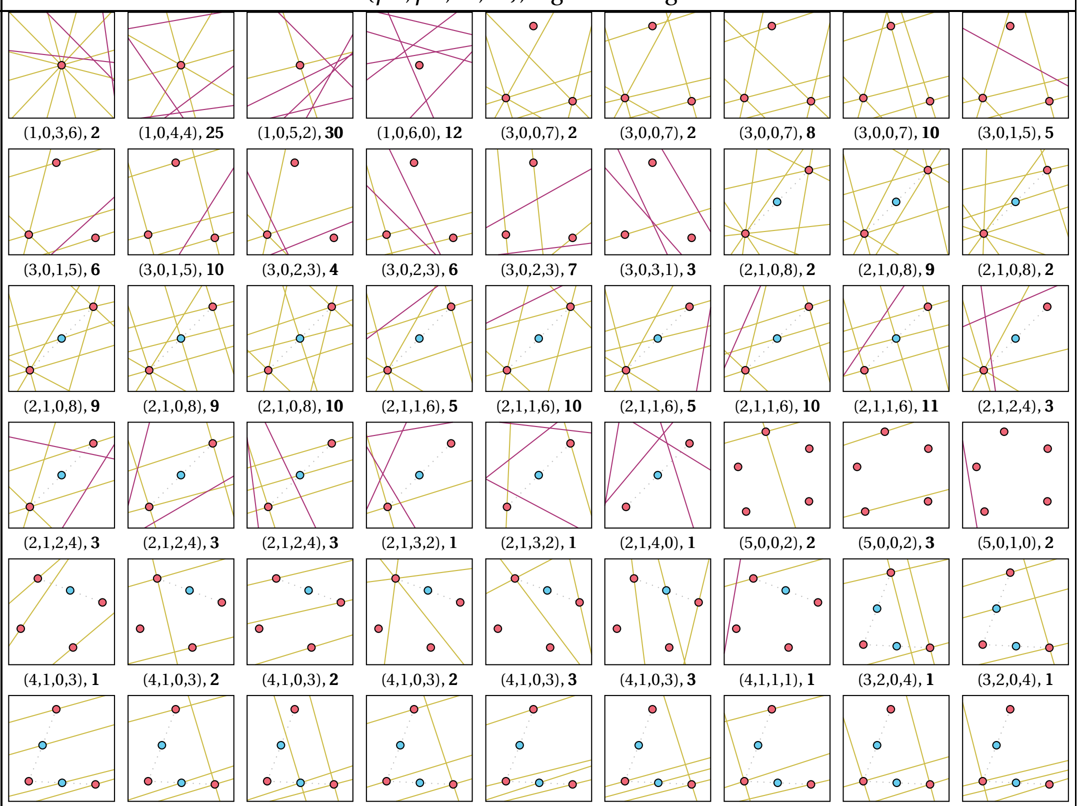
m	(p^f, p^d, l^f, l^a), algebraic degree								
3									
	(3,1,0,7), **6**	(3,1,0,7), **6**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,1,5), **1**
									
	(3,1,1,5), **1**	(3,1,1,5), **2**	(3,1,1,5), **2**	(3,1,1,5), **2**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**
									
	(3,1,1,5), **4**	(3,1,1,5), **4**	(3,1,1,5), **4**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**
									
	(3,1,2,3), **1**	(5,0,0,3), **2**	(5,0,0,3), **3**	(5,0,0,3), **4**	(5,0,1,1), **3**	(4,1,0,4), **1**	(4,1,0,4), **1**	(4,1,0,4), **1**	(4,1,0,4), **2**
									
	(4,1,0,4), **1**	(4,1,0,4), **2**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **2**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **3**
									
	(4,1,1,2), **1**	(4,1,1,2), **1**	(4,1,1,2), **2**	(4,1,1,2), **2**	(4,1,2,0), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**
									
	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**
									
	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(6,0,0,0), **3**
									
	(5,1,0,1), **1**	(5,1,0,1), **2**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **2**	(4,2,0,2), **1**
									
	(4,2,0,2), **1**	(4,2,1,0), **1**							

Table 8: Minimal problems with their associated degree.



m	(p^f, p^d, l^f, l^a), algebraic degree								
4	(1,0,3,6), **2**	(1,0,4,4), **25**	(1,0,5,2), **30**	(1,0,6,0), **12**	(3,0,0,7), **2**	(3,0,0,7), **2**	(3,0,0,7), **8**	(3,0,0,7), **10**	(3,0,1,5), **5**
	(3,0,1,5), **6**	(3,0,1,5), **10**	(3,0,2,3), **4**	(3,0,2,3), **6**	(3,0,2,3), **7**	(3,0,3,1), **3**	(2,1,0,8), **2**	(2,1,0,8), **9**	(2,1,0,8), **2**
	(2,1,0,8), **9**	(2,1,0,8), **9**	(2,1,0,8), **10**	(2,1,1,6), **5**	(2,1,1,6), **10**	(2,1,1,6), **5**	(2,1,1,6), **10**	(2,1,1,6), **11**	(2,1,2,4), **3**
	(2,1,2,4), **3**	(2,1,2,4), **3**	(2,1,2,4), **3**	(2,1,3,2), **1**	(2,1,3,2), **1**	(2,1,4,0), **1**	(5,0,0,2), **2**	(5,0,0,2), **3**	(5,0,1,0), **2**
	(4,1,0,3), **1**	(4,1,0,3), **2**	(4,1,0,3), **2**	(4,1,0,3), **2**	(4,1,0,3), **3**	(4,1,0,3), **3**	(4,1,1,1), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**
	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**

Table 9: Minimal problems with their associated degree.

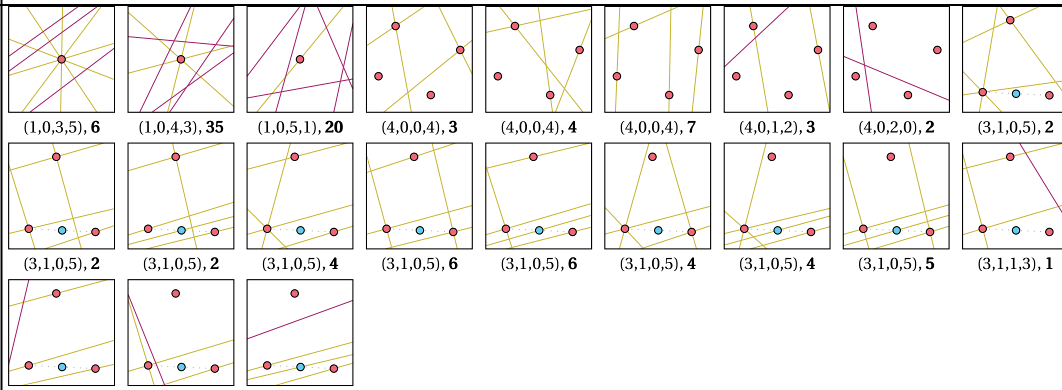
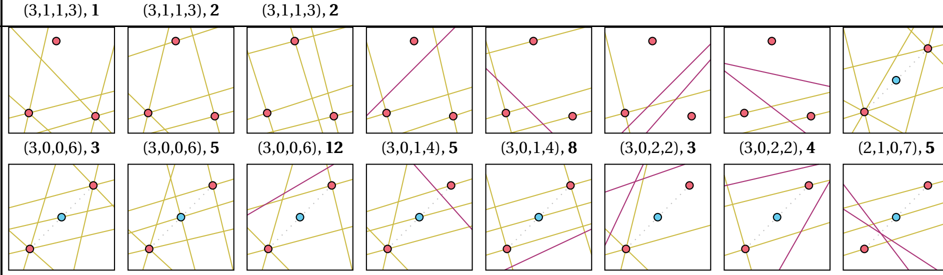
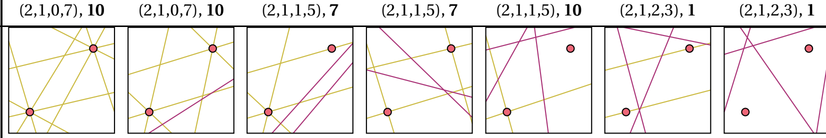
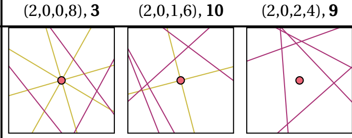
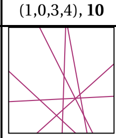
m	(p^f, p^d, l^f, l^a), algebraic degree
5	 (1,0,3,5), **6**; (1,0,4,3), **35**; (1,0,5,1), **20**; (4,0,0,4), **3**; (4,0,0,4), **4**; (4,0,0,4), **7**; (4,0,1,2), **3**; (4,0,2,0), **2**; (3,1,0,5), **2**; (3,1,0,5), **2**; (3,1,0,5), **2**; (3,1,0,5), **4**; (3,1,0,5), **6**; (3,1,0,5), **6**; (3,1,0,5), **4**; (3,1,0,5), **4**; (3,1,0,5), **5**; (3,1,1,3), **1**; (3,1,1,3), **1**; (3,1,1,3), **2**; (3,1,1,3), **2**
6	 (3,0,0,6), **3**; (3,0,0,6), **5**; (3,0,0,6), **12**; (3,0,1,4), **5**; (3,0,1,4), **8**; (3,0,2,2), **3**; (3,0,2,2), **4**; (2,1,0,7), **5**; (2,1,0,7), **5**; (2,1,0,7), **10**; (2,1,0,7), **10**; (2,1,1,5), **7**; (2,1,1,5), **7**; (2,1,1,5), **10**; (2,1,2,3), **1**; (2,1,2,3), **1**; (2,1,2,3), **1**
7	 (2,0,0,8), **3**; (2,0,1,6), **10**; (2,0,2,4), **9**; (2,0,2,4), **20**; (2,0,3,2), **6**; (2,0,3,2), **9**; (2,0,4,0), **3**
8	 (1,0,3,4), **10**; (1,0,4,2), **38**; (1,0,5,0), **8**
9	 (0,0,6,0), **114**

Table 10: Minimal problems with their associated degree.

Is the number of solutions an accurate complexity measure?

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A Galois-Theoretic Complexity Measure for Solving Systems of Algebraic Equations

Timothy Duff

March 25, 2025

Abstract

Motivated by applications of algebraic geometry, we introduce the *Galois width*, a quantity characterizing the complexity of solving algebraic equations in a restricted model of computation allowing only field arithmetic and adjoining polynomial roots. We explain why practical heuristics such as monodromy give (at least) lower bounds on this quantity, and discuss problems in geometry, optimization, statistics, and computer vision for which knowledge of the Galois width either leads to improvements over standard solution techniques or rules out this possibility entirely.

Galois width example

The Galois width of finding the roots of a univariate polynomial of degree n is

$$\begin{cases} 3 \quad , & \text{if } n = 4 \\ n \quad , & \text{else} \end{cases}$$

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$$\begin{cases} 3 & , \quad \text{if } n = 4 \\ n & , \quad \text{else} \end{cases}$$

The roots of a general quartic can be expressed in terms of the roots of its resolvent cubic and additional square roots thereof!

Galois width of vision minimal problems

Let 2 projective cameras take pictures of 7 points:

$$\Phi : \left((\mathbb{P}\,\mathbb{R}^{3\times 4})^2 \times (\mathbb{P}^3)^7 \right) / \mathrm{PGL}_4 \longrightarrow (\mathbb{P}^2)^7 \times (\mathbb{P}^2)^7$$

has generic fibers of size 3 and $\mathrm{GaloisWidth}(\Phi) = 3$.

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has generic fibers of size 3 and $\mathrm{GaloisWidth}(\Phi) = 3$.

Let 2 calibrated cameras take pictures of 5 points:

$$\Phi : \left((\mathrm{SO}(3) \times \mathbb{R}^3)^2 \times (\mathbb{P}^3)^5 \right) / G \longrightarrow (\mathbb{P}^2)^5 \times (\mathbb{P}^2)^5,$$
$$\text{where } G = \left\{ \left[\begin{smallmatrix} R & t \\ 0 & \lambda \end{smallmatrix}\right] \mid R \in \mathrm{SO}(3), t \in \mathbb{R}^3, \lambda \in \mathbb{R} \setminus \{0\} \right\},$$

has generic fibers of size 20 and $\mathrm{GaloisWidth}(\Phi) = 10$.

Order-One Rolling Shutter Cameras

joint with Marvin Hahn, Orlando Marigliano, Tomas Pajdla

Highlight @ CVPR 2025 :)

one of my long-term goals: algebra-geometry foundations of

rolling-shutter cameras:

take pictures by scanning across the scene, capturing the image row by row

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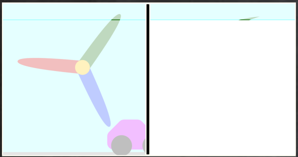

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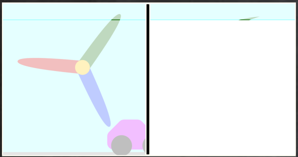

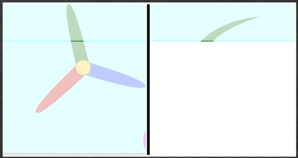

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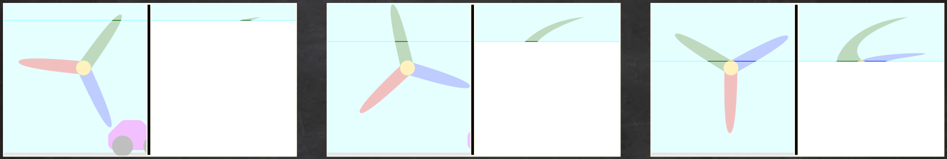

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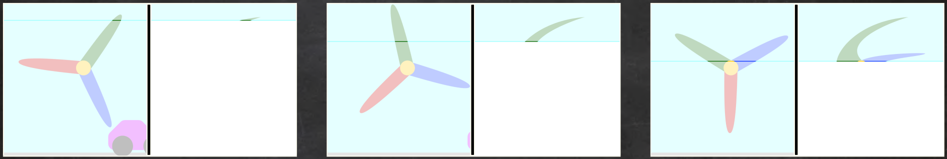

Algebraically:

- The image of a line is typically a higher-degree curve.

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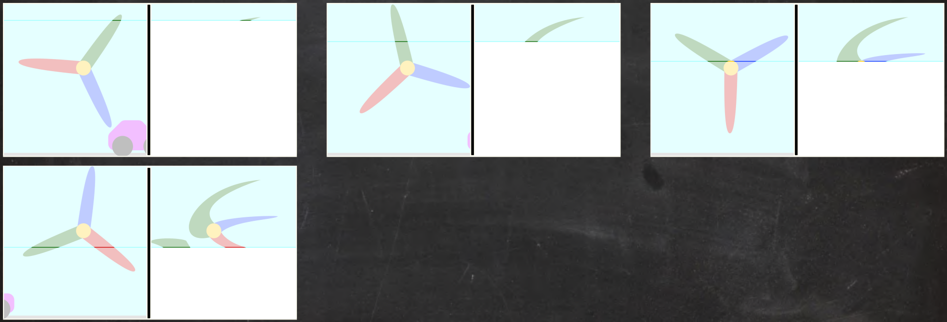

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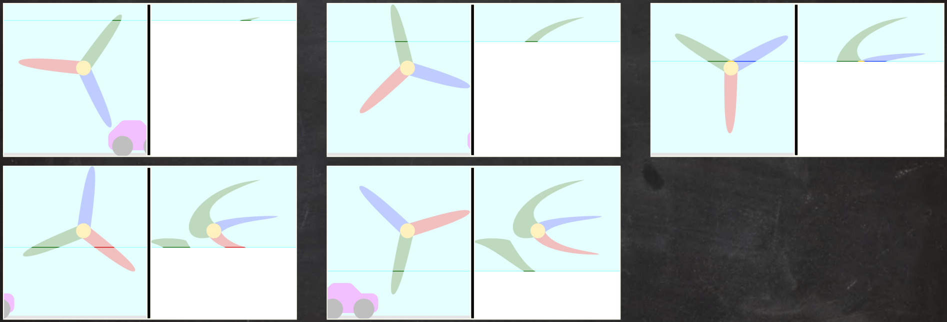

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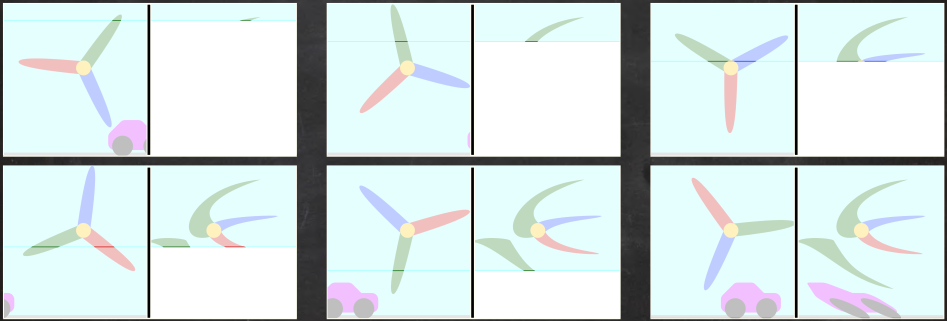

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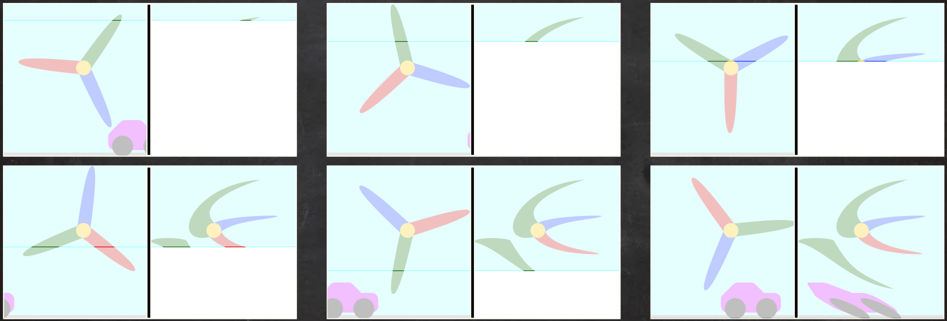
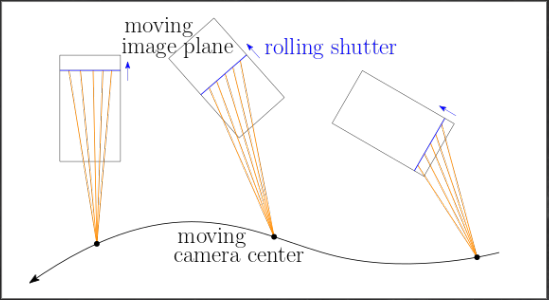


(by Cmglee @ Wikipedia
https://creativecommons.org/licenses/by-sa/3.0/deed.en
changes: added black separating line)

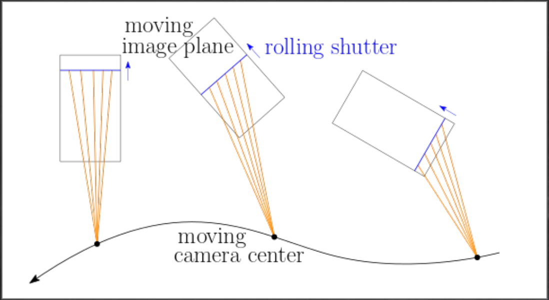
Algebraically:

- The image of a line is typically a higher-degree curve.
- A 3D point can appear more than once in the image.

Rolling-Shutter Camera



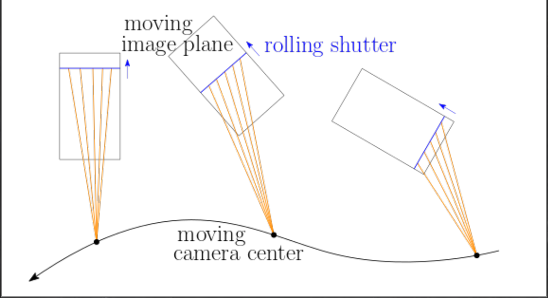
Rolling-Shutter Camera



Assume: rolling shutter parallel to y-axis on image plane:

$$
\begin{aligned}
\rho: \quad & \mathbb{P}^1 \longrightarrow (\mathbb{P}^2)^*, \\
& (v : t) \longmapsto (0 : 1 : 0) \vee (v : 0 : t) \equiv (-t : 0 : v).
\end{aligned}
$$

Rolling-Shutter Camera



On the affine chart $\{(v:t) \mid t \neq 0\} \subset \mathbb{P}^1$, the camera's position and orientation at time $\frac{v}{t}$ are

$$c(\tfrac{v}{t}) \in \mathbb{R}^3 \quad \text{and} \quad R(\tfrac{v}{t}) \in \mathrm{SO}(3).$$

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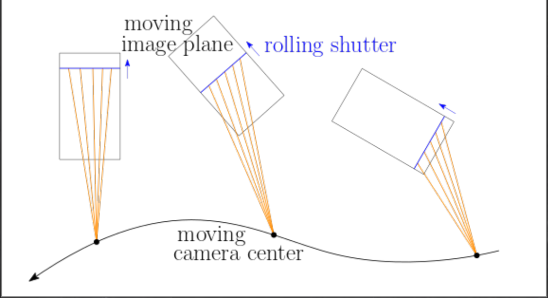

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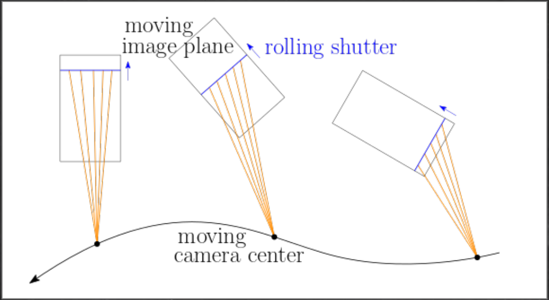
$$c(\tfrac{v}{t}) \in \mathbb{R}^3 \quad \text{and} \quad R(\tfrac{v}{t}) \in \mathrm{SO}(3).$$

Assume: c is a rational map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^3$.

How to take a picture?



At time $\frac{v}{t}$, the camera only observes a plane, not the whole ambient 3-space.

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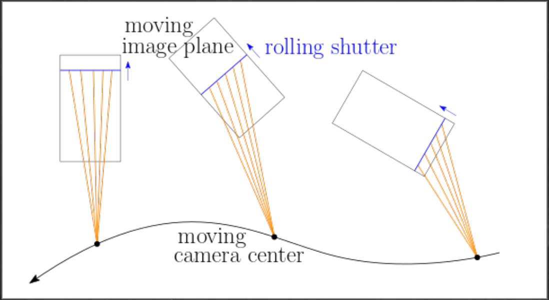

At time $\frac{v}{t}$, the camera only observes a plane, not the whole ambient 3-space. It maps that rolling plane onto the rolling shutter via the linear map given by

$$A(\tfrac{v}{t}) := R(\tfrac{v}{t}) \cdot [I_3 \mid -c(\tfrac{v}{t})].$$

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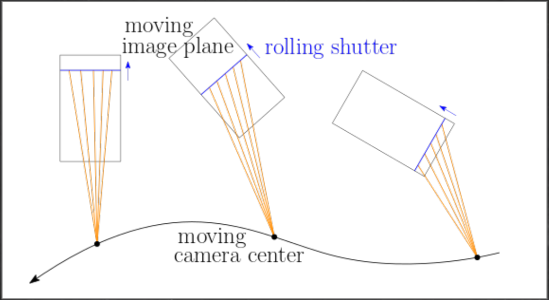

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$$A(\tfrac{v}{t}) := R(\tfrac{v}{t}) \cdot [I_3 \mid -c(\tfrac{v}{t})].$$

Hence, the rolling plane is the preimage of the rolling shutter under A:

$$\sigma(\tfrac{v}{t}) := (-t : 0 : v) \cdot A(\tfrac{v}{t}) \in (\mathbb{P}^3)^*.$$

How to take a picture?

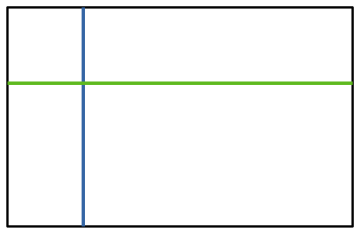

Image points are intersections of the rolling shutter with lines parallel to the x-axis:

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow (\mathbb{P}^2)^*, \\ (u:s) &\longmapsto (1:0:0) \vee (0:u:s) \equiv (0:-s:u) \end{aligned}$$

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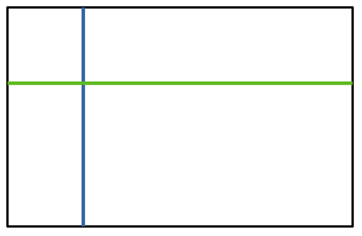

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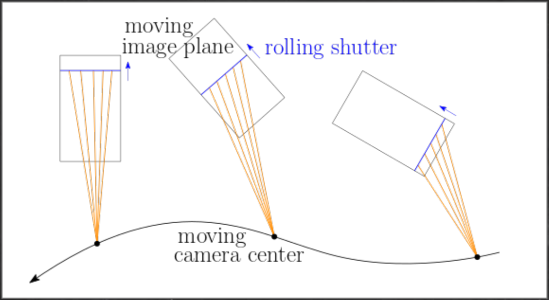
We think of the image plane as $\mathbb{P}^1 \times \mathbb{P}^1$ via the birational map

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^2, \\ ((v:t),(u:s)) &\mapsto (sv:ut:st). \end{aligned}$$

How to take a picture?



The camera ray mapping to the image point $((v:t),(u:s)) \in \mathbb{P}^1 \times \mathbb{P}^1$ is the point's preimage under $A(\frac{v}{t})$:

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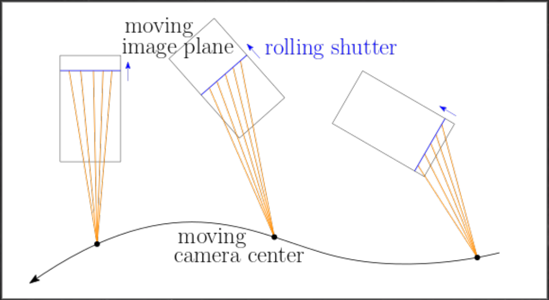

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$$\begin{aligned} \Lambda : \quad & \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathrm{Gr}(1,\mathbb{P}^3), \\ ((v:t),(u:s)) \mapsto & \underbrace{((-t:0:v)\cdot A(\tfrac{v}{t}))}_{\text{rolling plane } \sigma(\frac{v}{t})} \cap ((0:-s:u)\cdot A(\tfrac{v}{t})) \end{aligned}$$

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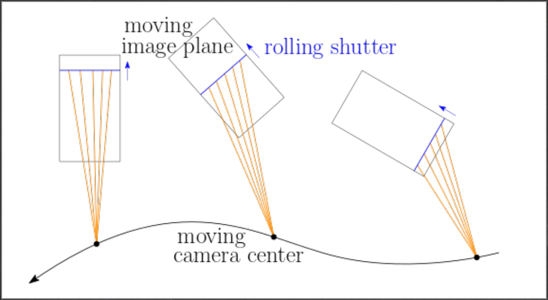

The camera ray mapping to the image point $((v : t), (u : s)) \in \mathbb{P}^1 \times \mathbb{P}^1$ is the point's preimage under $A(\frac{v}{t})$:

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Assume: Λ is rational

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The Zariski closure of the image of Λ is a surface $\mathcal{L}$ in $\mathrm{Gr}(1, \mathbb{P}^3)$, classically called a line congruence.

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Definition: The order of a line congruence $\mathcal{L} \subset \mathrm{Gr}(1, \mathbb{P}^3_{\mathbb{C}})$ is the number of lines on $\mathcal{L}$ that pass through a generic point in $\mathbb{P}^3_{\mathbb{C}}$.

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Observation: The number of times a generic point in $\mathbb{P}^3_{\mathbb{C}}$ is seen by a rolling-shutter camera is

$$\mathrm{order}(\overline{\mathrm{im}(\Lambda)}) \cdot \deg(\Lambda).$$

We call this the order of the camera.

Order-One Cameras

For a rolling-shutter camera of order one,

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1) the map Λ is birational onto its image $\mathcal{L} := \overline{\mathrm{im}(\Lambda)}$, i.e., its inverse $\Lambda^{-1} : \mathcal{L} \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ exists

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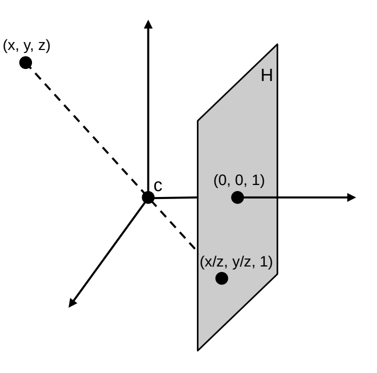
Observation: The picture-taking map is $\Lambda^{-1} \circ \Gamma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

Example: Global-Shutter Camera



$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2, X \mapsto AX$$

is a static rolling-shutter camera

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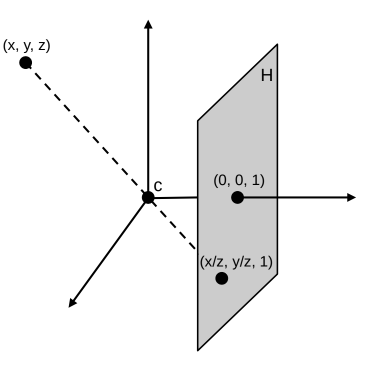

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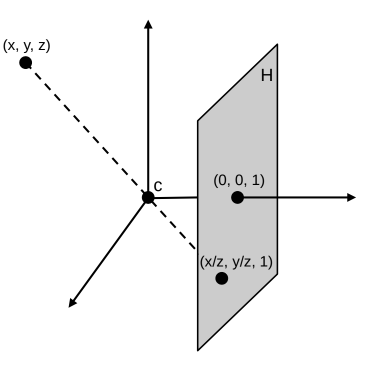

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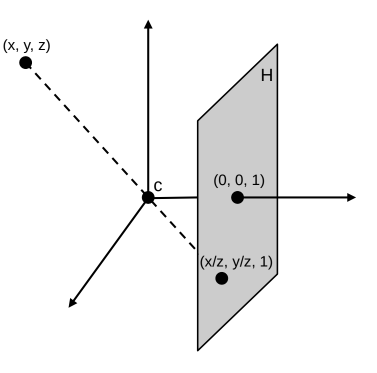

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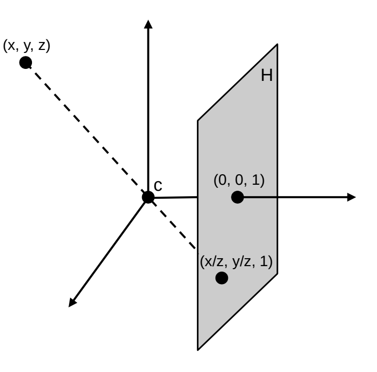

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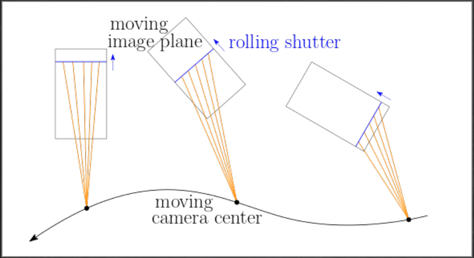
- Λ^{-1} intersects lines on $\mathcal{L}$ with image plane H

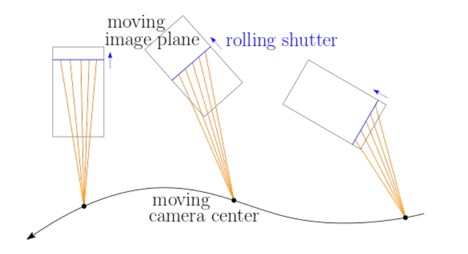
Order-One Cameras



Consider a rolling-shutter camera with camera-center map $c : \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ and rolling-planes map $\sigma : \mathbb{P}^1 \dashrightarrow (\mathbb{P}^3)^*$.

Theorem: The camera has order one if and only if

Order-One Cameras



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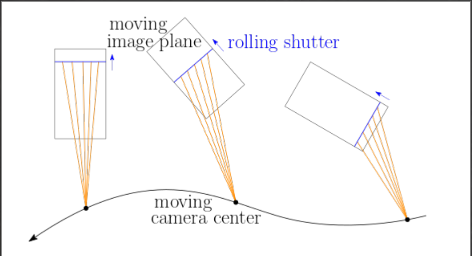

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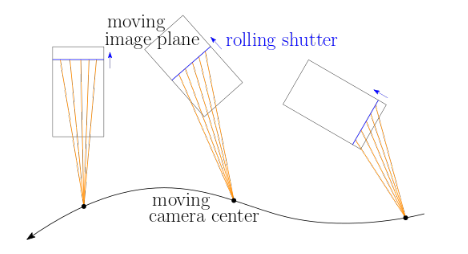

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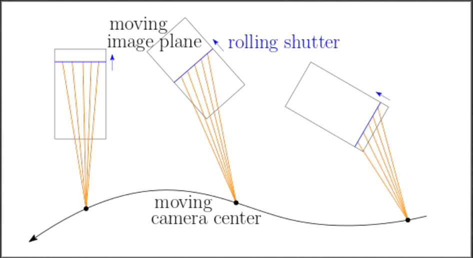
b) the rolling-planes map $\sigma : \mathbb{P}^1 \dashrightarrow K^\vee$ is birational,

c) and the center locus $C := \overline{\mathrm{im}(c)}$ is one of the following:

Order-One Cameras



Consider a rolling-shutter camera with camera-center map $c : \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ and rolling-planes map $\sigma : \mathbb{P}^1 \dashrightarrow (\mathbb{P}^3)^*$.

Theorem: The camera has order one if and only if

a) the intersection of all its rolling planes is a line K,

b) the rolling-planes map $\sigma : \mathbb{P}^1 \dashrightarrow K^\vee$ is birational,

c) and the center locus $C := \overline{\mathrm{im}(c)}$ is one of the following:

  I. C is a curve with $\#(K \cap C) = \deg(C) - 1$ (counted with multiplicities).

  II. $C = K$.

  III. C is a point on K.

Images of Lines

Recall: The picture-taking map is

$$\mathbb{P}^3 \overset{\Gamma}{\dashrightarrow} \mathcal{L} \overset{\Lambda^{-1}}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$

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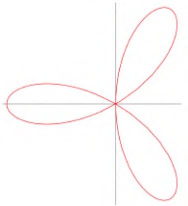
Theorem: The image of a generic line $L \subset \mathbb{P}^3$ under an order-one RS cameras is a curve of degree D that passes $D-1$ times (counted with multiplicity) through the point $(0 : 1 : 0)$.



$D = 4$

Images of Lines

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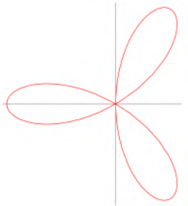
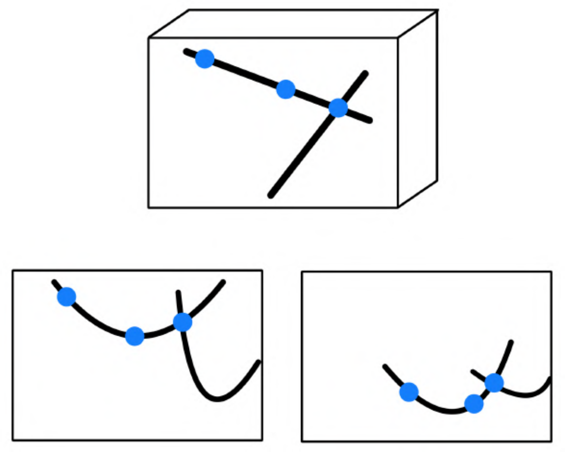


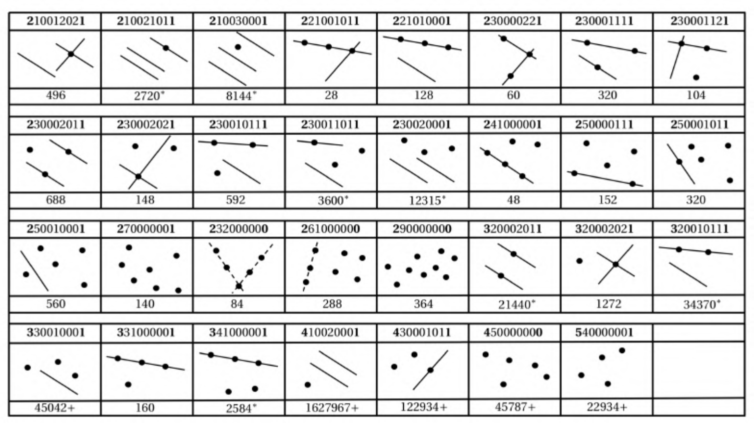
$D = 4$

Example: A order-one RS camera that moves along a line (with constant speed) and does not rotate maps lines to conics through a fixed point.

a point-line minimal problem of degre **28** for order-one RS cameras moving along a line with constant speed and no rotation



all complete-visibility point-line minimal problems for order-one RS cameras moving along a line with constant speed and no rotation



Now working one:

SfM & Triangulation of points & lines from Higher Order Rolling Shutter Cameras

joint with Petr Hruby, . . .

Open PhD Position in my group on Algebraic Geometry in Neural Network Theory !!!

machine learning	algebraic geometry
sample complexity & expressivity	dimension, degree, covering number
subnetworks & implicit bias	singularities
identifiability & hidden symmetries	fibers of the parametrization
optimization & gradient descent	critical point theory, discriminants, dynamical invariants

An Invitation to Neuroalgebraic Geometry

Giovanni Luca Marchetti[*1] Vahid Shahverdi[*1] Stefano Mereta[*1] Matthew Trager[*2] Kathlén Kohn[*1]

Abstract

In this expository work, we promote the study of function spaces parameterized by machine learning models through the lens of algebraic geometry. To this end, we focus on algebraic models, such as neural networks with polynomial activations, whose associated function spaces are semi-algebraic varieties. We outline a dictionary between algebro-geometric invariants of these varieties, such as dimension, degree, and singularities, and fundamental aspects of machine learning, such as sample complexity, expressivity, training dynamics, and implicit bias.

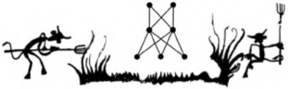


Figure 1. A neural variation of a celebrated doodle from the algebraic geometry literature (Grothendieck, 1968).