Minimal Problems in Computer Vision

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joint work with Timothy Duff (Georgia Tech),
Anton Leykin (Georgia Tech) & Tomas Pajdla (CTU in Prague)
Structure from Motion
Reconstruct 3D scenes and camera poses from 2D images

Rome in a Day: S. Agarwal, Y. Furukawa, N. Snavely, I. Simon, S. Seitz, R. Szeliski
Structure from Motion
Reconstruct 3D scenes and camera poses from 2D images

- Step 1: Identify common points and lines on given images
- Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses
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We use calibrated perspective cameras:

A camera is a matrix $C = [R \mid t]$, where $R \in SO(3)$ and $t \in \mathbb{R}^3$. 
Structure from Motion
Reconstruct 3D scenes and camera poses from 2D images

- **Step 1**: Identify common points and lines on given images

- **Step 2**: Reconstruct coordinates of 3D points and lines as well as camera poses

We use calibrated perspective cameras:
A camera is a matrix $C = [R \mid t]$, where $R \in \text{SO}(3)$ and $t \in \mathbb{R}^3$. Taking a picture of a point $x \in \mathbb{P}^3$: $x \mapsto Cx$
5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.
5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.

This problem has 20 solutions over $\mathbb{C}$ on generic input images.
(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)
Given 2 images of 5 points, recover 5 points in 3D and both camera poses.

This problem has **20 solutions** over $\mathbb{C}$ on generic input images. (Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

$\Rightarrow$ Since $0 < 20 < \infty$, the 5-Point-Problem is a **minimal** problem!
Another minimal problem

- Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses
Another minimal problem

- Given: 3 images of 3 points on a line, 1 attached line and 1 free line
- Recover: 3D coordinates of 3 points and 3 lines, 3 camera poses

This problem has 40 solutions over $\mathbb{C}$ on generic input images.
(solution = 3 camera poses and 3D coordinates of points and lines)

⇒ It is a minimal problem!
A **Point-Line-Problem (PLP)** consists of

- a number $m$ of cameras,
- a number $p$ of points,
- a number $\ell$ of lines,
- a set $\mathcal{I}$ of incidences between points and lines.
Minimal Problems

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- a number $m$ of cameras,
- a number $p$ of points,
- a number $\ell$ of lines,
- a set $\mathcal{I}$ of incidences between points and lines.

**Definition**

A PLP $(m, p, \ell, \mathcal{I})$ is **minimal** if, given $m$ generic 2D-arrangements each consisting of $p$ points and $\ell$ lines satisfying the incidences $\mathcal{I}$, it has a positive and finite number of solutions over $\mathbb{C}$.

(solution = $m$ camera poses and 3D coordinates of $p$ points and $\ell$ lines satisfying the incidences $\mathcal{I}$)
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Can we list all minimal PLP$s$?
How many solutions do they have?
### Minimal PLPs

<table>
<thead>
<tr>
<th>$m$ views</th>
<th>$p^f p^d l^f l^a_\alpha$</th>
<th>(p, l, I)</th>
<th>Minimal Degree</th>
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*VI - XII*
Joint camera map

(3D-arrangement of $p$ points and $\ell$ lines satisfying incidences $I$, $\text{cam}_1, \ldots, \text{cam}_m$)

Lemma
If a PLP is minimal, then $\dim(X) + \dim(C) = \dim(Y)$. 

VII - XII
Joint camera map

(3D-arrangement, $\text{cam}_1, \ldots, \text{cam}_m) \mapsto (2D-\text{arr}_1, \ldots, 2D-\text{arr}_m)

of $p$ points and $\ell$ lines satisfying incidences $\mathcal{I}$
Joint camera map

$$\mathcal{X} \times C \rightarrow \mathcal{Y}$$

(3D-arrangement of $p$ points and $\ell$ lines satisfying incidences $I$)

$$\mathcal{X} = \{ (X_1, \ldots, X_p, L_1, \ldots, L_\ell) \in (\mathbb{P}^3)^p \times (\text{Grass}(1, 3))^\ell \}$$

$$\mathcal{Y} = \{ (x_1, \ldots, x_m, l_1, \ldots, l_m) \in (\mathbb{P}^2)^{mp \times m\ell} \}$$

$$C = \{ [R_1|t_1], \ldots, [R_m|t_m] \}$$

Lemma: If a PLP is minimal, then

$$\dim(\mathcal{X}) + \dim(C) = \dim(\mathcal{Y})$$
Joint camera map
\[ \mathcal{X} \times C \longrightarrow Y \]
where \( \mathcal{X} \) is the 3D-arrangement of points and \( C \) is the set of cameras \( \text{cam}_1, \ldots, \text{cam}_m \) mapping to \( (2D-\text{arr}_1, \ldots, 2D-\text{arr}_m) \) in the 2D-arrangement.

- \( \mathbb{P}^n \) is the \( n \)-dimensional projective space.
- \( \mathbb{G}_{1,n} = \{ \text{lines in } \mathbb{P}^n \} \) is the Grassmannian of lines in \( \mathbb{P}^n \).
- \( \mathcal{X} = \{ (X_1, \ldots, X_p, L_1, \ldots, L_\ell) \in (\mathbb{P}^3)^p \times (\mathbb{G}_{1,3})^\ell \mid \forall (i,j) \in \mathcal{I} : X_i \in L_j \} \)

**Lemma**
If a PLP is minimal, then \( \dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(Y) \).
Joint camera map

\[ \mathcal{X} \times \mathcal{C} \rightarrow \mathcal{Y} \]

(3D-arrangement, \( \text{cam}_1, \ldots, \text{cam}_m \)) \rightarrow (2D-arr_1, \ldots, 2D-arr_m)

of \( p \) points and \( \ell \) lines satisfying incidences \( \mathcal{I} \)

- \( \mathbb{P}^n = n \)-dimensional projective space
- \( \mathbb{G}_{1,n} = \{ \text{lines in } \mathbb{P}^n \} = \text{Grassmannian of lines in } \mathbb{P}^n \)
- \( \mathcal{X} = \{(X_1, \ldots, X_p, L_1, \ldots, L_\ell) \in (\mathbb{P}^3)^p \times (\mathbb{G}_{1,3})^\ell \mid \forall (i, j) \in \mathcal{I} : X_i \in L_j\} \)
- \( \mathcal{Y} = \left\{(x_{1,1}, \ldots, x_{m,p}, l_{1,1}, \ldots, l_{m,\ell}) \in (\mathbb{P}^2)^{mp} \times (\mathbb{G}_{1,2})^{m\ell} \mid \forall (i, j) \in \mathcal{I} : x_{k,i} \in l_{k,j}\right\} \)

Lemma
If a PLP is minimal, then \( \dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y}) \).
Joint camera map

$$\mathcal{X} \times C \longrightarrow \mathcal{Y}$$

(3D-arrangement, cam$_1$, \ldots, cam$_m$) $\mapsto$ (2D-arr$_1$, \ldots, 2D-arr$_m$)

of $p$ points and $\ell$ lines satisfying incidences $\mathcal{I}$

- $\mathbb{P}^n = n$-dimensional projective space
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- $C = \left\{ ([R_1|t_1], \ldots, [R_m|t_m]) \mid \forall i = 1, \ldots, m : R_i \in \text{SO}(3), t_i \in \mathbb{R}^3, \right. \left. R_1 = I_3, t_1 = 0, t_{2,1} = 1 \right\}$

Lemma

If a PLP is minimal, then $\dim(\mathcal{X}) + \dim(C) = \dim(\mathcal{Y})$.
Joint camera map

\[
\begin{align*}
\mathcal{X} \times \mathcal{C} & \to \mathcal{Y} \\
(3\text{-}arr, \text{ cam}_1, \ldots, \text{ cam}_m) & \mapsto (2\text{-}arr_1, \ldots, 2\text{-}arr_m)
\end{align*}
\]

of \(p\) points and \(\ell\) lines satisfying incidences \(I\)

- \(\mathbb{P}^n = n\)-dimensional projective space
- \(\mathbb{G}_{1,n} = \{\text{lines in } \mathbb{P}^n\} = \text{Grassmannian of lines in } \mathbb{P}^n\)
- \(\mathcal{X} = \{(X_1, \ldots, X_p, L_1, \ldots, L_\ell) \in (\mathbb{P}^3)^p \times (\mathbb{G}_{1,3})^\ell | \forall (i, j) \in I : X_i \in L_j\}\)
- \(\mathcal{Y} = \left\{(x_{1,1}, \ldots, x_{m,p}, l_{1,1}, \ldots, l_{m,\ell}) \in (\mathbb{P}^2)^{mp} \times (\mathbb{G}_{1,2})^{m\ell} | \forall (i, j) \in I : x_{k,i} \in l_{k,j}\right\}\)
- \(\mathcal{C} = \left\{([R_1|t_1], \ldots, [R_m|t_m]) | \forall i = 1, \ldots, m : R_i \in \text{SO}(3), t_i \in \mathbb{R}^3,\right.\left.\begin{align*}
R_1 &= I_3, \quad t_1 = 0, \quad t_{2,1} = 1
\end{align*}\right\}\)

Lemma

If a PLP is minimal, then \(\dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y})\).
Algebraic varieties

Definition
A \textit{variety} is the common zero set of a system of polynomial equations.

A variety looks like a manifold \textit{almost everywhere}:
Algebraic varieties

Definition
A variety is the common zero set of a system of polynomial equations.

A variety looks like a manifold almost everywhere:

Definition
A variety is irreducible if it is not the union of two proper subvarieties.
Algebraic varieties

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A variety looks like a manifold almost everywhere:

Definition
A variety is irreducible if it is not the union of two proper subvarieties. The dimension of an irreducible variety is its local dimension as a manifold.
Algebraic varieties

Definition
A **variety** is the common zero set of a system of polynomial equations.

A variety looks like a manifold *almost everywhere*:

![Variety diagrams](image)

Definition
A variety is **irreducible** if it is not the union of two proper subvarieties. The **dimension** of an irreducible variety is its local dimension as a manifold.

\( \mathcal{X}, \mathcal{C} \) and \( \mathcal{Y} \) are irreducible varieties!
Deriving the big table

\[ \mathcal{X} \times \mathcal{C} \longrightarrow \mathcal{Y} \]

(3D-arrangement of \( p \) points and \( \ell \) lines with incidences \( \mathcal{I} \))

Lemma

If a PLP is minimal, then \( \dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y}) \).
Deriving the big table

\[ \mathcal{X} \times \mathcal{C} \rightarrow \mathcal{Y} \]

(3D-arrangement \( \text{cam}_1, \ldots, \text{cam}_m \)) \( \mapsto \) (2D-arr\(_1\), \ldots, 2D-arr\(_m\))

of \( p \) points and \( \ell \) lines with incidences \( I \)

**Lemma**

If a PLP is minimal, then \( \dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y}) \).

**Theorem**

- If \( m > 6 \), then \( \dim(\mathcal{X}) + \dim(\mathcal{C}) \neq \dim(\mathcal{Y}) \).
Deriving the big table

\[ \mathcal{X} \times \mathcal{C} \rightarrow \mathcal{Y} \]

(3D-arrangement, \( \text{cam}_1, \ldots, \text{cam}_m \)) \[\rightarrow\] (2D-arr\(_1\), \ldots, 2D-arr\(_m\))

of \( p \) points and \( \ell \) lines with incidences \( \mathcal{I} \)

**Lemma**

If a PLP is minimal, then \( \dim(\mathcal{X}') + \dim(\mathcal{C}) = \dim(\mathcal{Y}) \).

**Theorem**

- If \( m > 6 \), then \( \dim(\mathcal{X}') + \dim(\mathcal{C}) \neq \dim(\mathcal{Y}) \).
- There are exactly 39 PLPs with \( \dim(\mathcal{X}') + \dim(\mathcal{C}) = \dim(\mathcal{Y}) \):

<table>
<thead>
<tr>
<th>( m ) views</th>
<th>( p \times \text{proj} )</th>
<th>( \mathcal{I} )</th>
<th>Minimal Degree</th>
</tr>
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<tr>
<td>6</td>
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<td>Y</td>
<td>&gt; 450(_k)</td>
</tr>
<tr>
<td>1013(_t)</td>
<td>N</td>
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<td></td>
</tr>
<tr>
<td>1005(_t)</td>
<td>N</td>
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<td></td>
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<td>2011(_t)</td>
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<td></td>
</tr>
<tr>
<td>2003(_t)</td>
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<tr>
<td>2003(_t)</td>
<td>Y</td>
<td></td>
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<tr>
<td>2003(_t)</td>
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</table>

Y: Yellow, N: Green, \# represents a specific number.
Deriving the big table

\[ \mathcal{X} \times C \rightarrow Y \]

(3D-arrangement, \( \text{cam}_1, \ldots, \text{cam}_m \)) \( \mapsto \) (2D-arr\(_1\), \ldots, 2D-arr\(_m\))

Lemma

A PLP with \( \dim(\mathcal{X}) + \dim(C) = \dim(Y) \) is minimal if and only if its joint camera map \( \mathcal{X} \times C \rightarrow Y \) is dominant.
Deriving the big table

\( \mathcal{X} \times \mathcal{C} \longrightarrow \mathcal{Y} \)

(3D-arrangement of \( p \) points and \( \ell \) lines satisfying incidences \( \mathcal{I} \))

**Lemma**

A PLP with \( \dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y}) \) is minimal if and only if its joint camera map \( \mathcal{X} \times \mathcal{C} \rightarrow \mathcal{Y} \) is dominant.

**Definition**

A map \( \varphi : A \rightarrow B \) is **surjective** if for every \( b \in B \) there is an \( a \in A \) such that \( \varphi(a) = b \).

**Definition**

A map \( \varphi : A \rightarrow B \) is **dominant** if for almost every \( b \in B \) there is an \( a \in A \) such that \( \varphi(a) = b \).
Deriving the big table

\[ X \times \mathcal{C} \rightarrow Y \]

of \( p \) points and \( \ell \) lines
satisfying incidences \( I \)

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A PLP with \( \dim(X) + \dim(C) = \dim(Y) \) is minimal if and only if its joint camera map \( X \times C \rightarrow Y \) is dominant.

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**Fact**

A map \( \varphi : A \rightarrow B \) between irreducible varieties \( A \) and \( B \) is dominant if and only if for almost every \( a \in A \) the differential \( D_a \varphi : T_a A \rightarrow T_{\varphi(a)} B \) is surjective.
Deriving the big table

\[
\begin{align*}
\mathcal{X} \times \mathcal{C} \rightarrow \mathcal{Y} \\
(3D\text{-arrangement of } p \text{ points and } \ell \text{ lines}) & \mapsto (2D\text{-arr}_1, \ldots, 2D\text{-arr}_m)
\end{align*}
\]

Lemma

A PLP with \( \dim(\mathcal{X}) + \dim(\mathcal{C}) = \dim(\mathcal{Y}) \) is minimal if and only if its joint camera map \( \mathcal{X} \times \mathcal{C} \rightarrow \mathcal{Y} \) is dominant.

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Can check this computationally! It is only linear algebra!
For $m \in \{2, 3\}$: compute number of solutions with Gröbner bases (standard technique in algebraic geometry).

For $m \in \{4, 5, 6\}$: compute number of solutions with homotopy continuation and monodromy (state-of-the-art method in numerical algebraic geometry).

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XI - XII
For $m \in \{2, 3\}$: compute number of solutions with **Gröbner bases** (standard technique in algebraic geometry)

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For $m \in \{2, 3\}$: compute number of solutions with Gr"obner bases (standard technique in algebraic geometry)

For $m \in \{4, 5, 6\}$: compute number of solutions with homotopy continuation and monodromy (state-of-the-art method in numerical algebraic geometry)
Monodromy

Pick random \((X_0, C_0)\) \(\in X \times C\)

Set \(Y = \Phi(X_0, C_0)\)

Pick \(Y' \in Y\) along a random path from \(Y\) to \(Y'\)

Track the solution \((X_0, C_0)\) for \(Y\) to a solution \((X', C'_0)\) for \(Y'\) via homotopy continuation

Along a random path from \(Y'\) to \(Y\)

Track the solution \((X', C'_0)\) for \(Y'\) to a solution \((X_1, C_1)\) for \(Y\) via homotopy continuation

Keep on circulating between \(Y\) and \(Y'\) until no more solutions for \(Y\) are found
Monodromy

- Pick random \((X_0, C_0) \in \mathcal{X} \times \mathcal{C}\)
- Set \(Y = \Phi(X_0, C_0)\)
- Pick \(Y' \in \mathcal{Y}\)
Monodromy

- Pick random \((X_0, C_0) \in \mathcal{X} \times \mathcal{C}\)
- Set \(Y = \Phi(X_0, C_0)\)
- Pick \(Y' \in \mathcal{Y}\)
- Along a random path from \(Y\) to \(Y'\) track the solution \((X_0, C_0)\) for \(Y\) to a solution \((X'_0, C'_0)\) for \(Y'\) via **homotopy continuation**
Monodromy

- Pick random \((X_0, C_0) \in X \times C\)
- Set \(Y = \Phi(X_0, C_0)\)
- Pick \(Y' \in \mathcal{Y}\)
- Along a random path from \(Y\) to \(Y'\), track the solution \((X_0, C_0)\) for \(Y\) to a solution \((X'_0, C'_0)\) for \(Y'\) via **homotopy continuation**
- Along a random path from \(Y'\) to \(Y\), track the solution \((X'_0, C'_0)\) for \(Y'\) to a solution \((X_1, C_1)\) for \(Y\) via **homotopy continuation**
**Monodromy**

- Pick random \((X_0, C_0) \in X \times C\)
- Set \(Y = \Phi(X_0, C_0)\)
- Pick \(Y' \in Y\)
- Along a random path from \(Y\) to \(Y'\) track the solution \((X_0, C_0)\) for \(Y\) to a solution \((X'_0, C'_0)\) for \(Y'\) via **homotopy continuation**
- Along a random path from \(Y'\) to \(Y\) track the solution \((X'_0, C'_0)\) for \(Y'\) to a solution \((X_1, C_1)\) for \(Y\) via **homotopy continuation**
- Keep on circulating between \(Y\) and \(Y'\) until no more solutions for \(Y\) are found
Monodromy

- Pick random \((X_0, C_0) \in \mathcal{X} \times \mathcal{C}\)
- Set \(Y = \Phi(X_0, C_0)\)
- Pick \(Y' \in \mathcal{Y}\)
- Along a random path from \(Y\) to \(Y'\) track the solution \((X_0, C_0)\) for \(Y\) to a solution \((X'_0, C'_0)\) for \(Y'\) via **homotopy continuation**
- Along a random path from \(Y'\) to \(Y\) track the solution \((X'_0, C'_0)\) for \(Y'\) to a solution \((X_1, C_1)\) for \(Y\) via **homotopy continuation**
- Keep on circulating between \(Y\) and \(Y'\) until no more solutions for \(Y\) are found
Monodromy

- Pick random \((X_0, C_0) \in \mathcal{X} \times \mathcal{C}\)
- Set \(Y = \Phi(X_0, C_0)\)
- Pick \(Y' \in \mathcal{Y}\)
- Along a random path from \(Y\) to \(Y'\)
  track the solution \((X_0, C_0)\) for \(Y\)
  to a solution \((X'_0, C'_0)\) for \(Y'\)
  via **homotopy continuation**
- Along a random path from \(Y'\) to \(Y\)
  track the solution \((X'_0, C'_0)\) for \(Y'\)
  to a solution \((X_1, C_1)\) for \(Y\)
  via **homotopy continuation**
- Keep on circulating between \(Y\) and \(Y'\)
  until no more solutions for \(Y\) are found
Pick random \((X_0, C_0) \in X \times C\)

Set \(Y = \Phi(X_0, C_0)\)

Pick \(Y' \in \mathcal{Y}\)

Along a random path from \(Y\) to \(Y'\) track the solution \((X_0, C_0)\) for \(Y\) to a solution \((X'_0, C'_0)\) for \(Y'\) via **homotopy continuation**

Along a random path from \(Y'\) to \(Y\) track the solution \((X'_0, C'_0)\) for \(Y'\) to a solution \((X_1, C_1)\) for \(Y\) via **homotopy continuation**

Keep on circulating between \(Y\) and \(Y'\) until no more solutions for \(Y\) are found
Thanks for your attention!