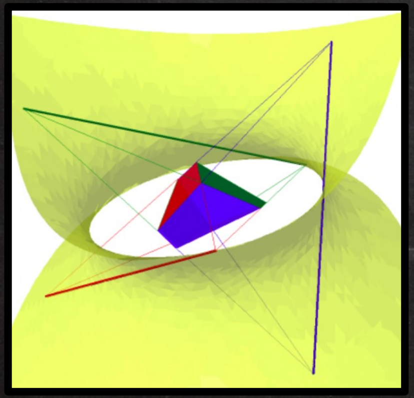
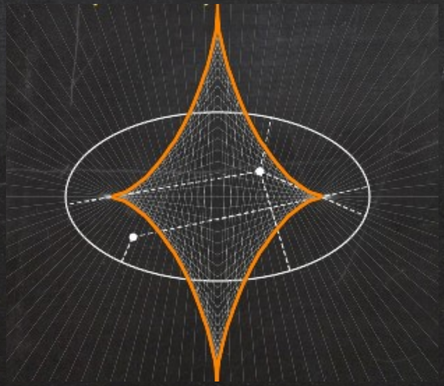
Nonlinear Algebra of Data Science & AI

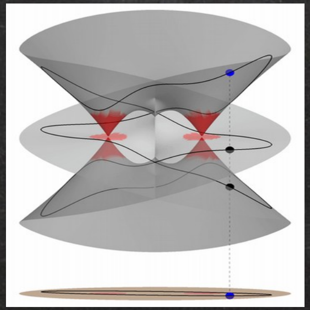
Kathlén Kohn

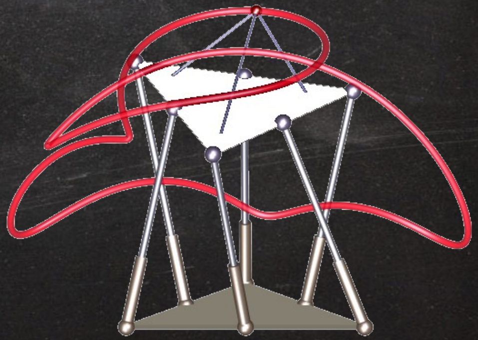
KTH VETENSKAP OCH KONST

WASP WALLENBERG AUTONOMOUS SYSTEMS AND SOFTWARE PROGRAM









Linear algebra

All undergraduate students learn about Gaussian elimination, a general method for solving linear systems of algebraic equations:

Input:

$$x + 2y + 3z = 5$$
$$7x + 11y + 13z = 17$$
$$19x + 23y + 29z = 31$$

Output:

$$x = -35/18$$
$$y = 2/9$$
$$z = 13/6$$

Solving very large linear systems is central to applied mathematics.

Nonlinear algebra

Lucky students also learn about Gröbner bases, a general method for non-linear systems of algebraic equations:

Input:

$$x^2 + y^2 + z^2 = 2$$
$$x^3 + y^3 + z^3 = 3$$
$$x^4 + y^4 + z^4 = 4$$

Output:

$$3z^{12} - 12z^{10} - 12z^9 + 12z^8 + 72z^7 - 66z^6 - 12z^4 + 12z^3 - 1 = 0$$

$$4y^2 + (36z^{11} + 54z^{10} - 69z^9 - 252z^8 - 216z^7 + 573z^6 + 72z^5 - 12z^4 - 99z^3 + 10z + 3)\, y + 36z^{11} + 48z^{10} - 72z^9 - 234z^8 - 192z^7 + 564z^6 - 48z^5 + 96z^4 - 96z^3 + 10z^2 + 8 = 0$$

$$4x + 4y + 36z^{11} + 54z^{10} - 69z^9 - 252z^8 - 216z^7 + 573z^6 + 72z^5 - 12z^4 - 99z^3 + 10z + 3 = 0$$

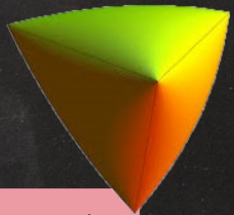
This is very hard for large systems, but . . .

The world is non-linear!

Many models in the sciences and engineering are characterized by polynomial equations. Such a set is an algebraic variety.

- Algebraic statistics
- Machine learning
- Optimization
- Computer vision
- Robotics
- Complexity theory
- Cryptography
- Biology
- Economics
- ...

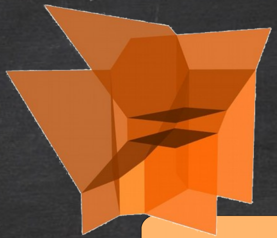
Nonlinear Algebra



Combinatorics

Convex Geometry

Algebraic Geometry +



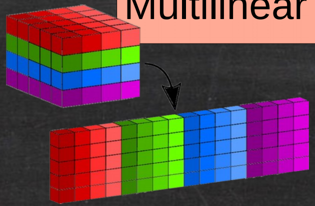
Discrete Geometry

Tropical Geometry

Algebraic Topology

Representation Theory

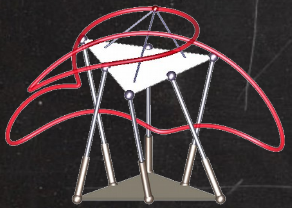
Multilinear Algebra



Numerics

...

Number Theory



Applications

Algebraic Statistics

Invariant theory and scaling algorithms for maximum likelihood estimation
arXiv: 2003.13662

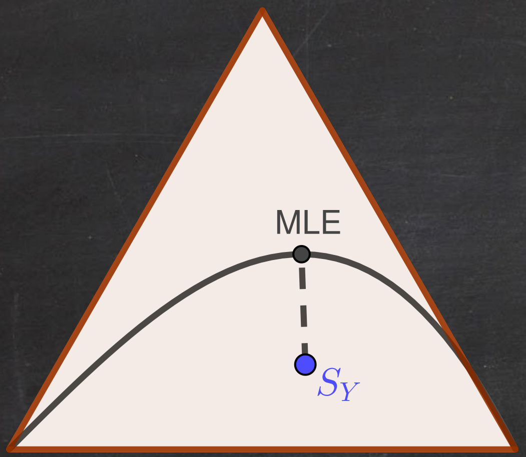
joint work with

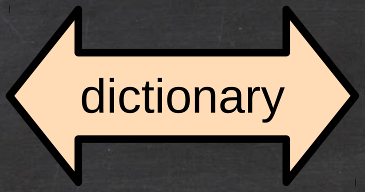
Carlos Améndola
(TU Munich / Univ. Ulm)

Philipp Reichenbach
(TU Berlin)

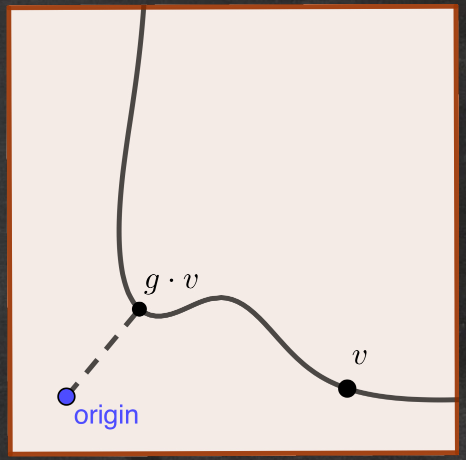
Anna Seigal
(University of Oxford)

Statistics





Invariant theory



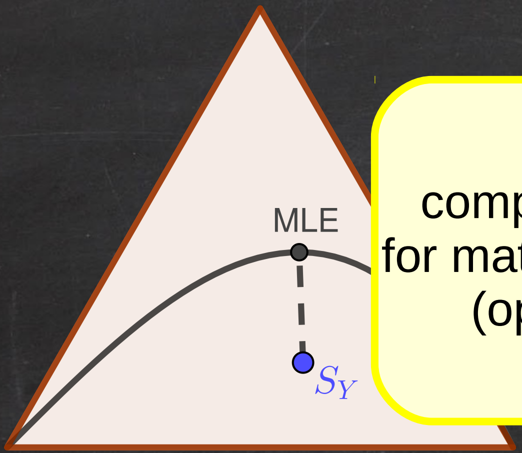
Given: statistical model
sample data S_Y

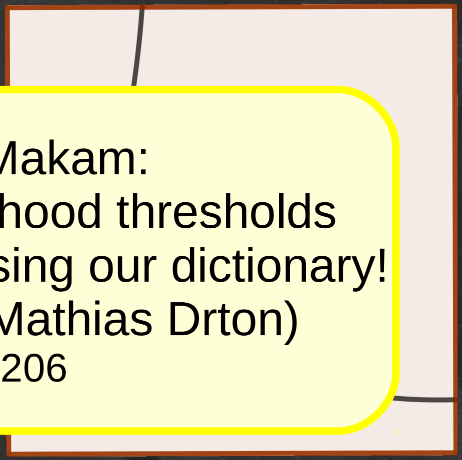
Task: find maximum likelihood estimate (MLE)
= distribution in model that best fits S_Y

Given: orbit G•v = {g•v | g ϵ G}
of a group G acting on vector v

Task: compute capacity
= closest distance of orbit to origin

Statistics | **Invariant theory**





H. Derksen, V. Makam:
computed maximum likelihood thresholds
for matrix normal models using our dictionary!
(open question due to Mathias Drton)
arXiv: 2007.10206

Convergence analysis ⟷ Null cone (Hilbert, 1893); Stability notions

historical progression ↑ (Statistics) | historical progression ↓ (Invariant theory)

Algorithms to find MLE (1940) ⟷ Algorithms for capacity / null cone membership testing (2017 – now)

Algebraic Statistics

Projective geometry of Wachspress coordinates
arXiv: 1904.02123

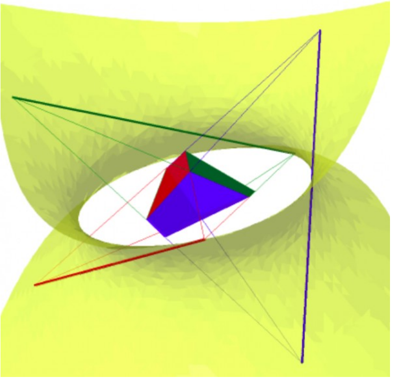
Moment Varieties of Measures on Polytopes
arXiv: 1807.10258

joint works with

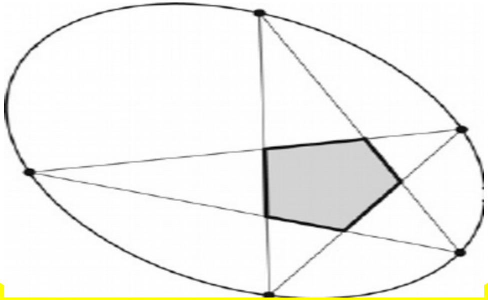
Kristian Ranestad
(University of Oslo)

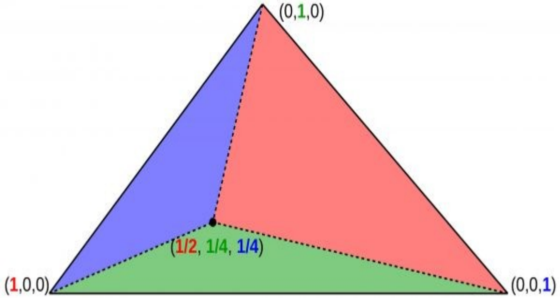
Boris Shapiro
(Stockholm University)

Bernd Sturmfels
(MPI MiS Leipzig / UC Berkeley)

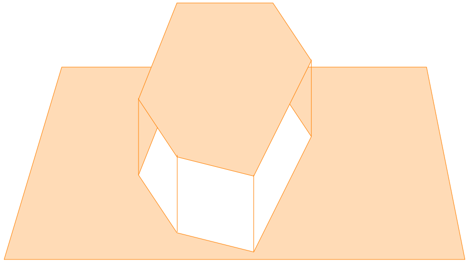


classical algebraic geometry
adjoint hypersurfaces

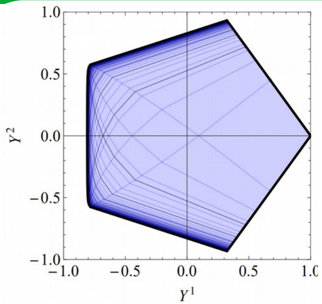




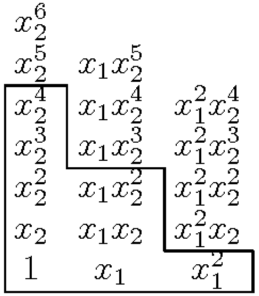
geometric modeling
barycentric coordinates
for arbitrary polytopes



algebraic statistics:
moments of uniform distributions
on polytopes



physics
scattering amplitudes



intersection theory:
Segre classes of
monomial schemes

Machine Learning

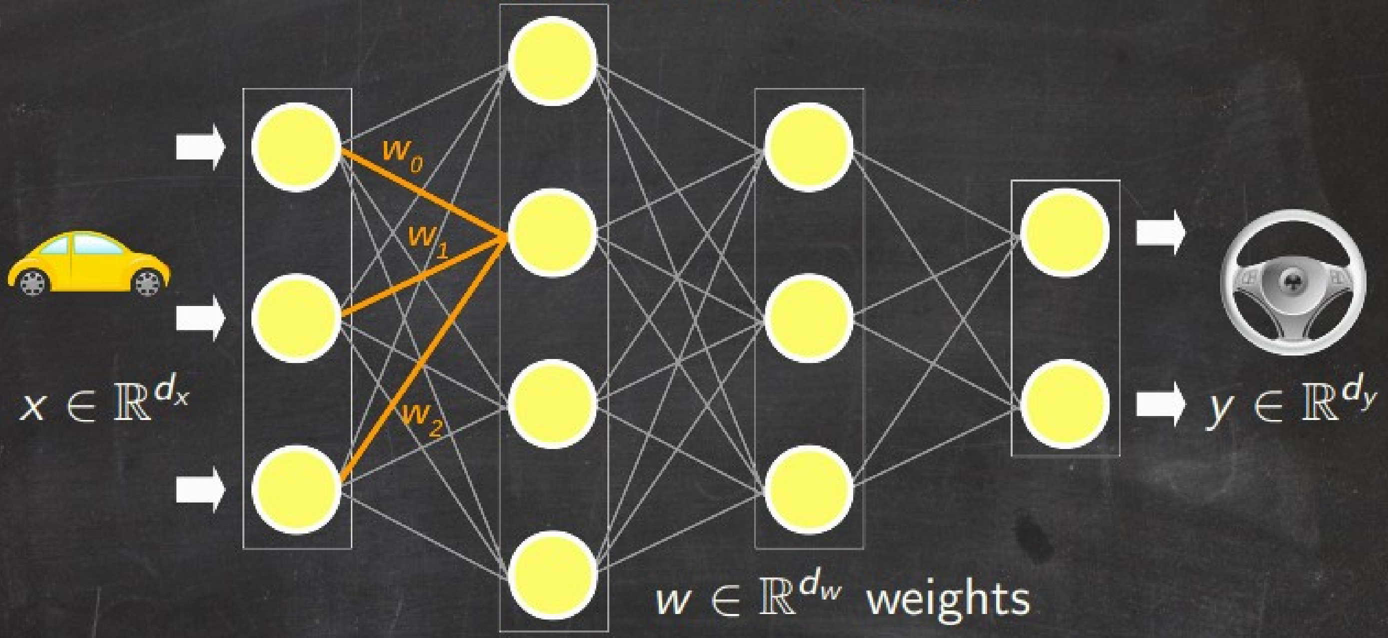
Pure and Spurious Critical Points: a Geometric Study of Linear Networks
arXiv: 1910.01671

joint work with

Matthew Trager
(Amazon AI)

Joan Bruna
(Courant Institute NYU)

Neural Networks



A neural network is defined by a continuous mapping $\Phi : \mathbb{R}^{d_w} \times \mathbb{R}^{d_x} \longrightarrow \mathbb{R}^{d_y}$.

Definition $\mathcal{M}_\Phi := \left\{ \Phi(w, \cdot) : \mathbb{R}^{d_x} \to \mathbb{R}^{d_y} \mid w \in \mathbb{R}^{d_w} \right\} \subset C(\mathbb{R}^{d_x}, \mathbb{R}^{d_y})$

is called the **neuromanifold** of Φ.

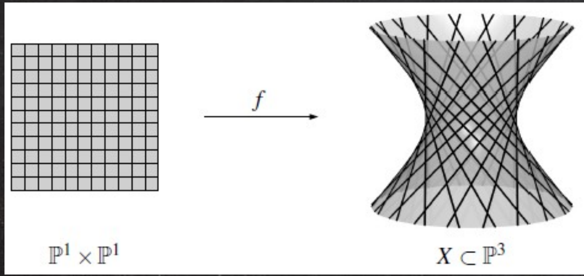
Observation
1. Φ piecewise smooth $\Rightarrow \mathcal{M}_\Phi$ manifold with singularities
2. $\dim \mathcal{M}_\Phi \leq d_w$

The neuromanifold of the linear network Φ is

$$\mathcal{M}_\Phi = \left\{ M \in \mathbb{R}^{d_h \times d_0} \mid \mathrm{rk}(M) \leq \underbrace{\min\{d_0, d_1, \ldots, d_h\}}_{=: r} \right\}.$$

width of layers

This is a determinantal variety!



machine learning with quadratic loss

⟷

minimizing Euclidean distance to determinantal variety

for linear networks with smooth convex losses:

	quadratic loss	other loss
r=min{d_0,d_h}	no bad min.	no bad min.
r<min{d_0,d_h}	no bad min.	**bad min.**

convex optimization on vector space (← r=min{d_0,d_h} row)

special embedding of determinantal varieties (↑ quadratic loss column)

Computer Vision

PL1P – Point-line Minimal Problems under Partial Visibility in Three Views
arXiv: 2003.05015

PLMP – Point-Line Minimal Problems in Complete Multi-View Visibility
arXiv: 1903.10008

joint works with

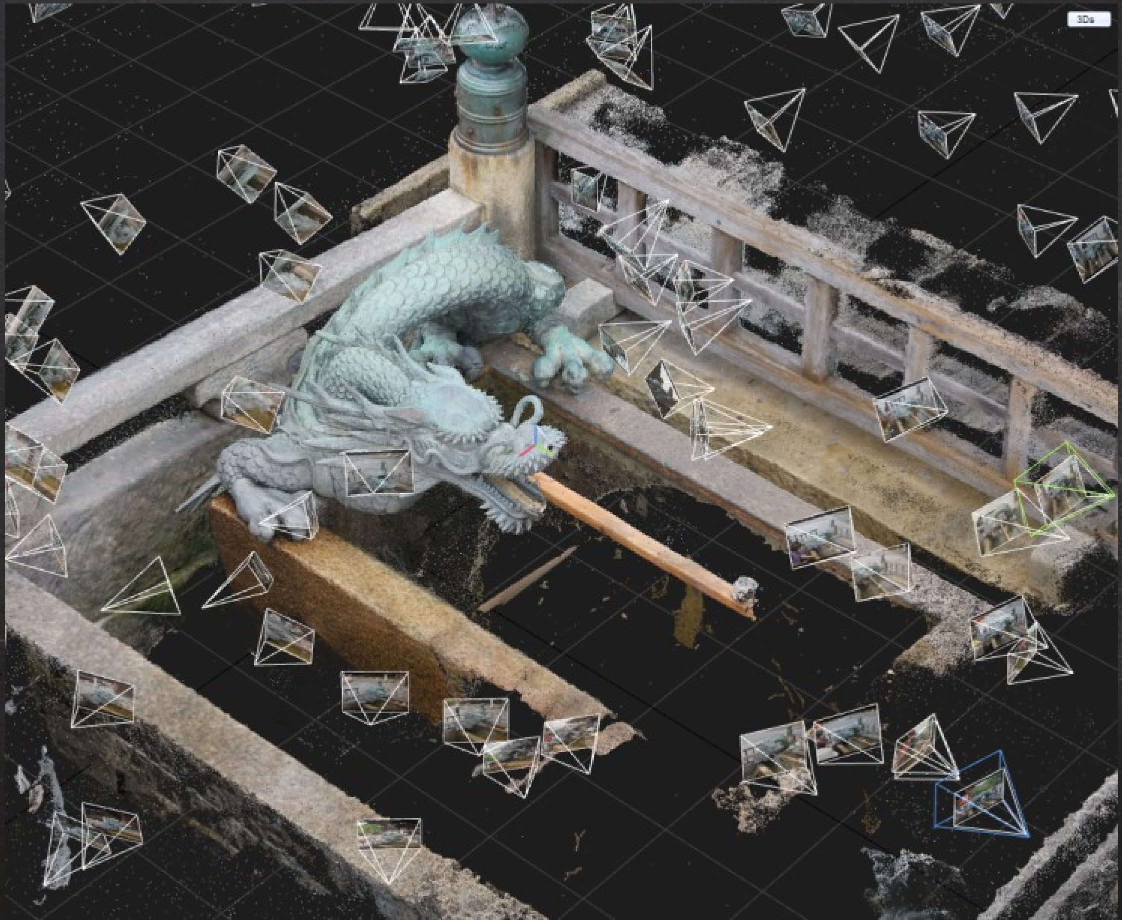
Timothy Duff
(Georgia Tech)

Anton Leykin
(Georgia Tech)

Tomas Pajdla
(CIIRC CTU in Prague)

Goal:

Reconstruct 3D scenes and camera poses from 2D images



3D Reconstruction Pipeline

Input images

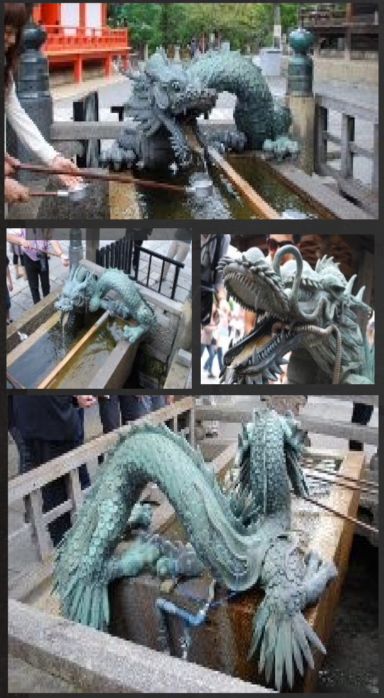
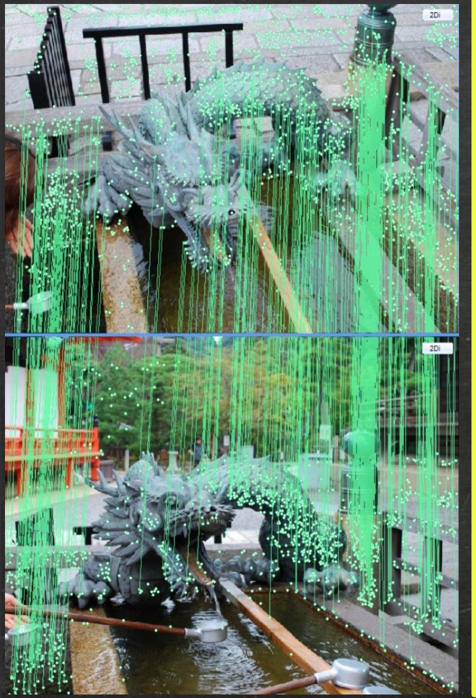
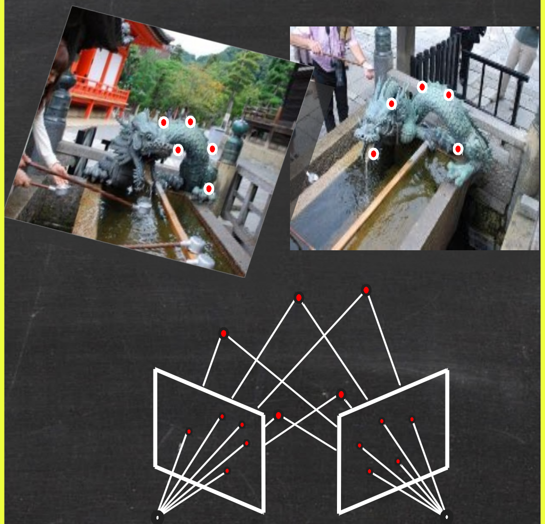


Image Matching



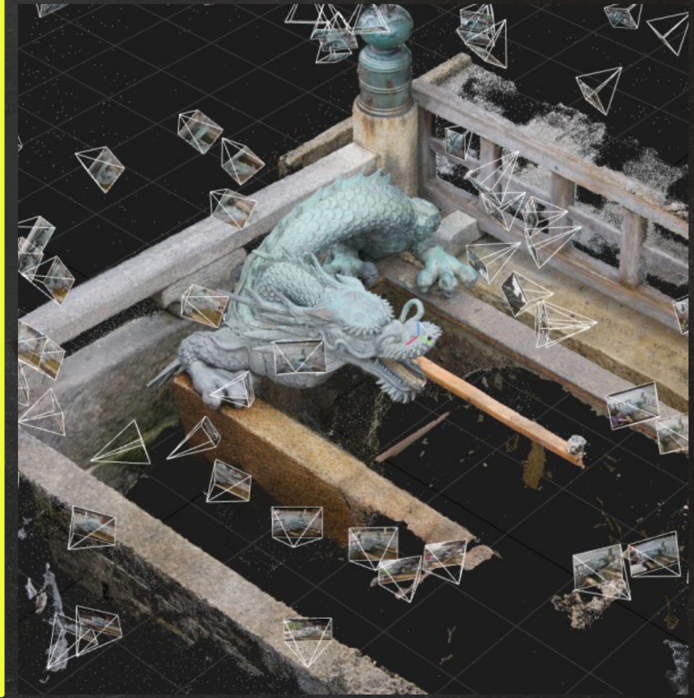
Identify common points and lines on given images

Camera Geometry



Reconstruct 3D points and lines as well as camera poses

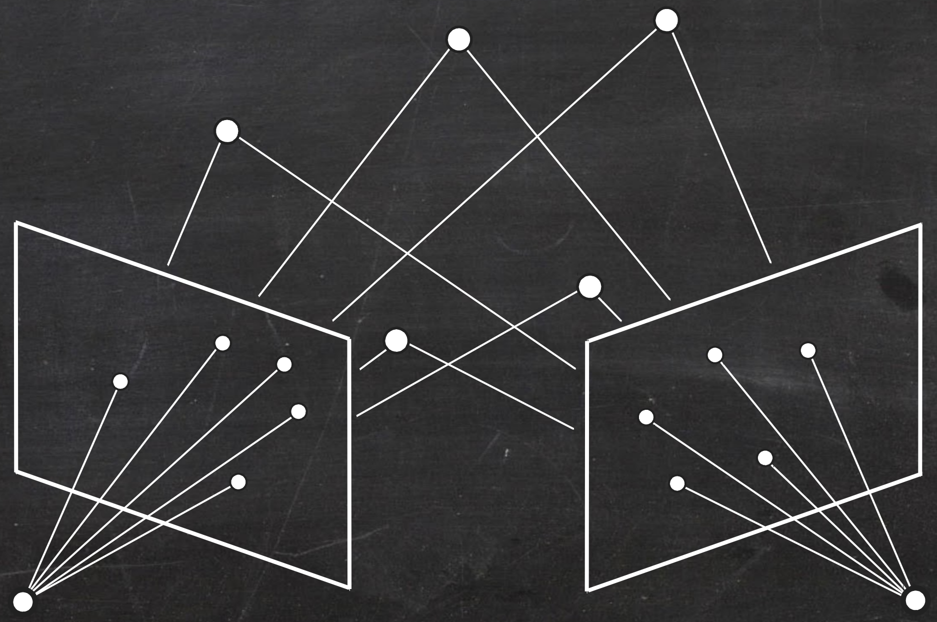
Cameras & points



This is an **algebraic** problem!

Example: The 5-Point Problem

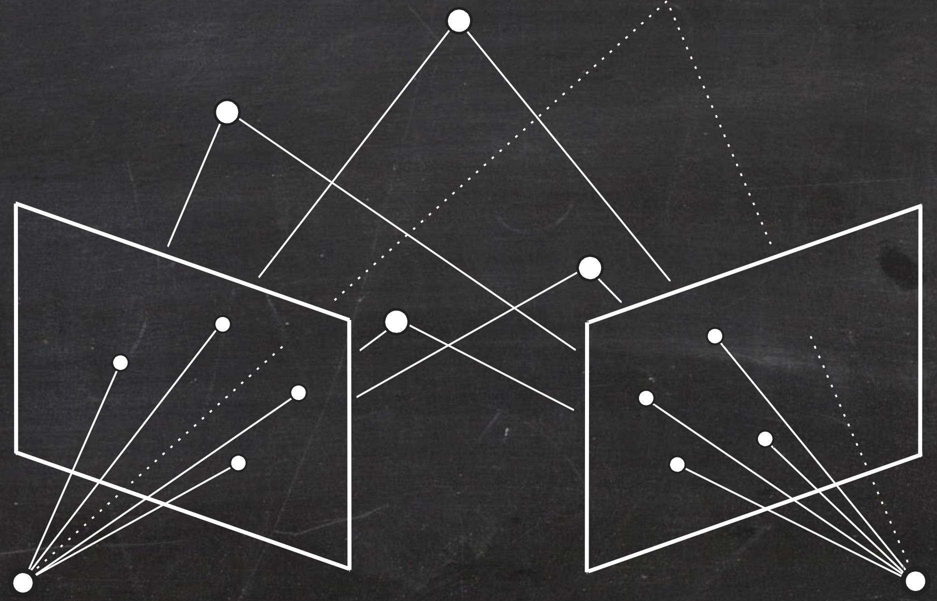
- Given: 2 images showing **5** points
- Goal: recover **5** points in 3D, and both (relative) camera poses



This problem has 20 solutions for generic input images
(counted over the complex numbers).

An Underconstrained Problem

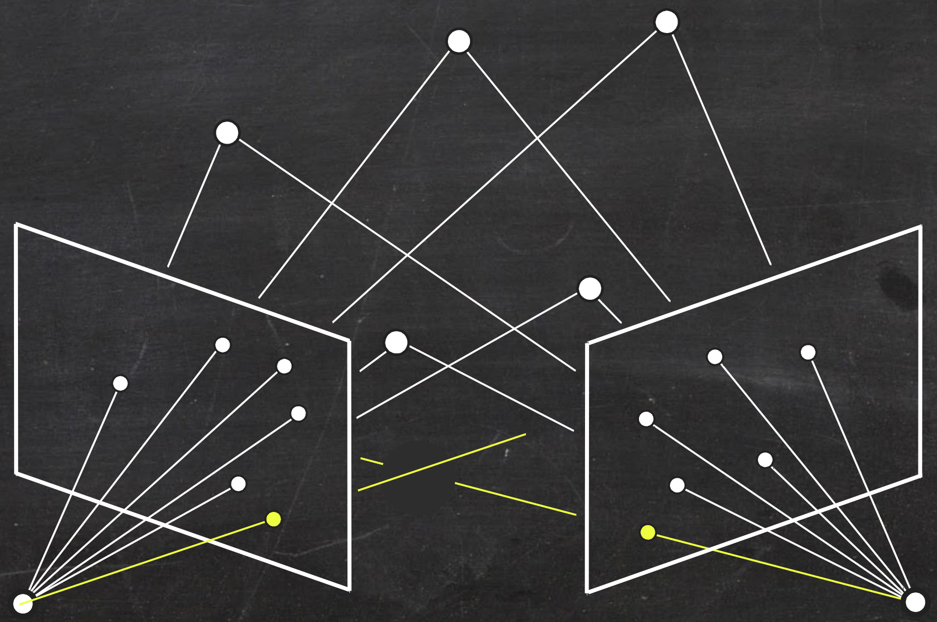
- Given: 2 images showing **4** points
- Goal: recover **4** points in 3D, and both (relative) camera poses



This problem has infinitely many solutions for generic input images.

An Overconstrained Problem

- Given: 2 images showing **6** points
- Goal: recover **6** points in 3D, and both (relative) camera poses



This problem has 0 solutions for generic input images.

Some input images have solutions, but they are **not stable under noise** in the input images!

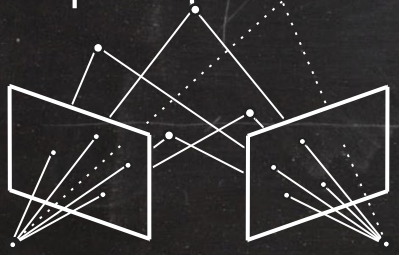
Minimal Problems

Definition: A 3D reconstruction problem is **minimal** if

$$0 < \# \text{ solutions} < \infty$$

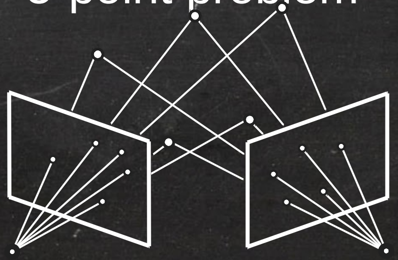
for generic (random) input images.

4-point problem



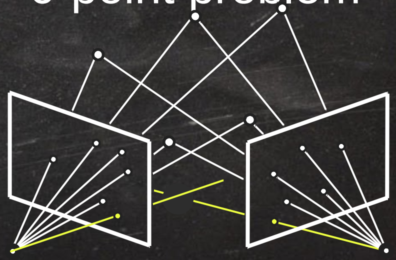
∞ solutions
not minimal

5-point problem



20 solutions
minimal

6-point problem

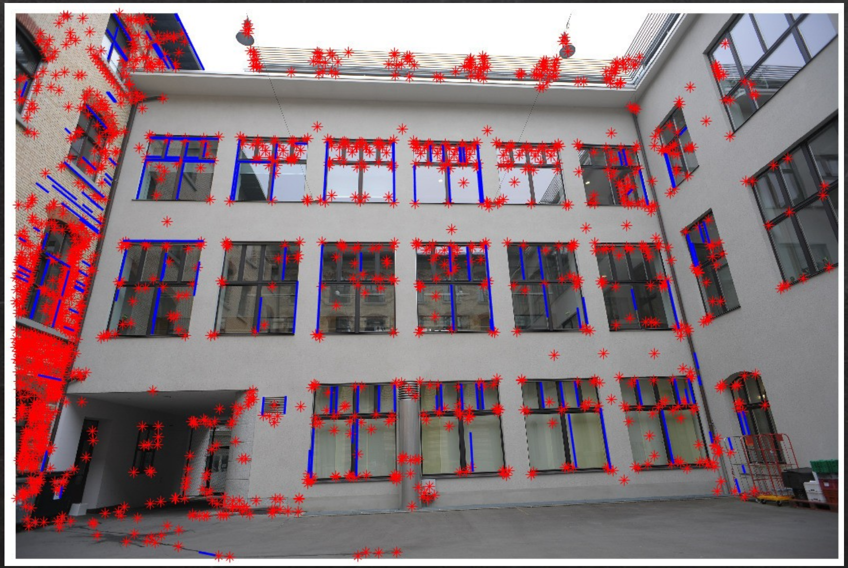


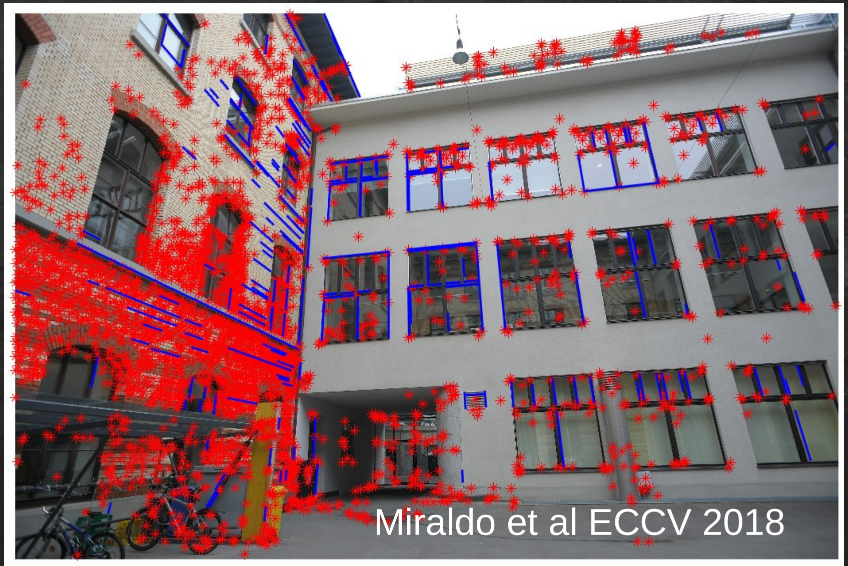
0 solutions
not minimal

Fundamental Research Questions

1. Can we list **all** minimal problems?
2. How many solutions do they have?

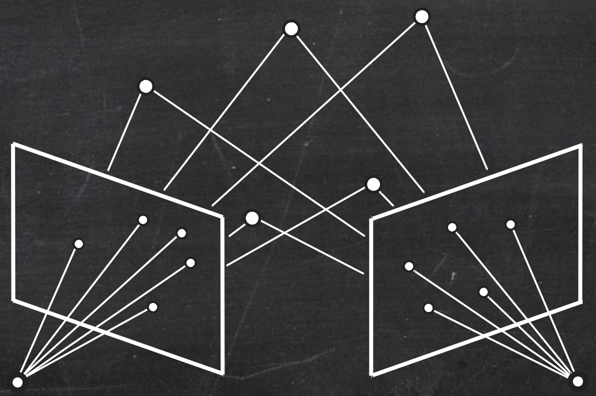
We do not only want to work with points,
but also with lines and their incidences!





Our Result

We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image.



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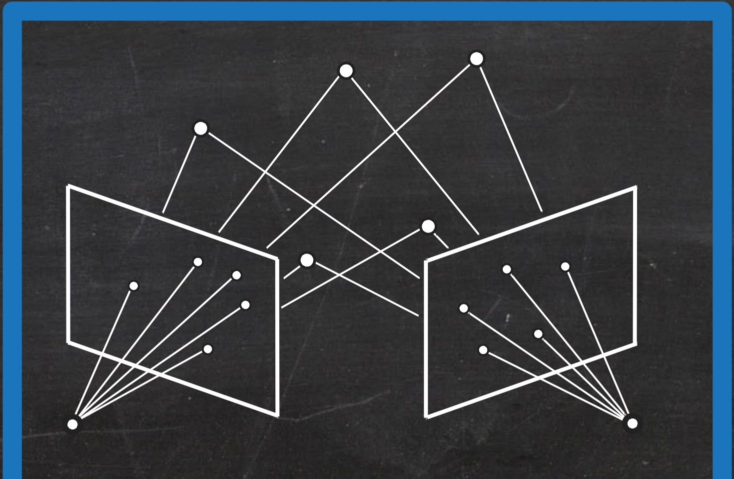
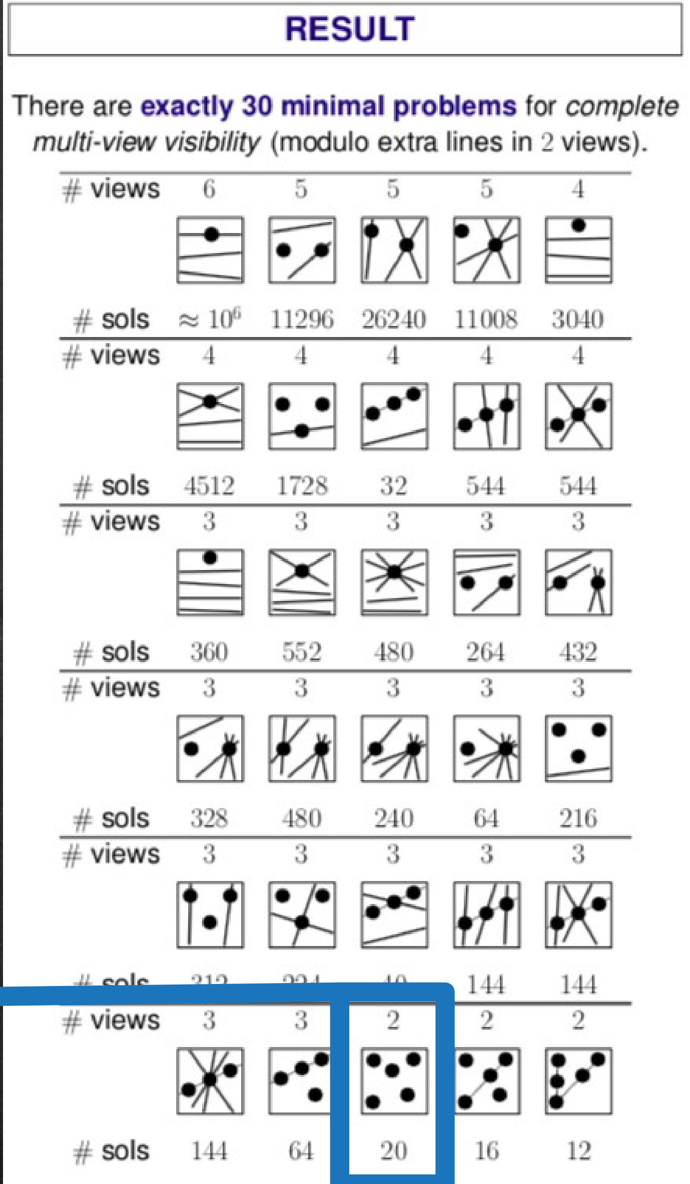


The 5-point problem has 20 solutions.



RESULT

There are **exactly 30 minimal problems** for *complete multi-view visibility* (modulo extra lines in 2 views).

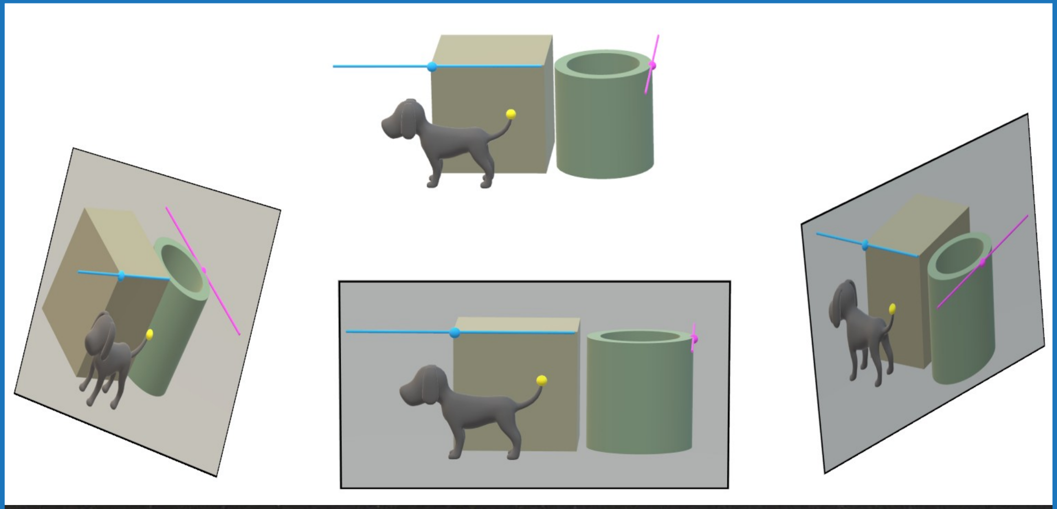
# views	6	5	5	5	4
# sols	≈ 10^6	11296	26240	11008	3040
# views	4	4	4	4	4
# sols	4512	1728	32	544	544
# views	3	3	3	3	3
# sols	360	552	480	264	432
# views	3	3	3	3	3
# sols	328	480	240	64	216
# views	3	3	3	3	3
# sols	[illegible]	[illegible]	[illegible]	144	144
# views	3	3	2	2	2
# sols	144	64	20	16	12

Our Result

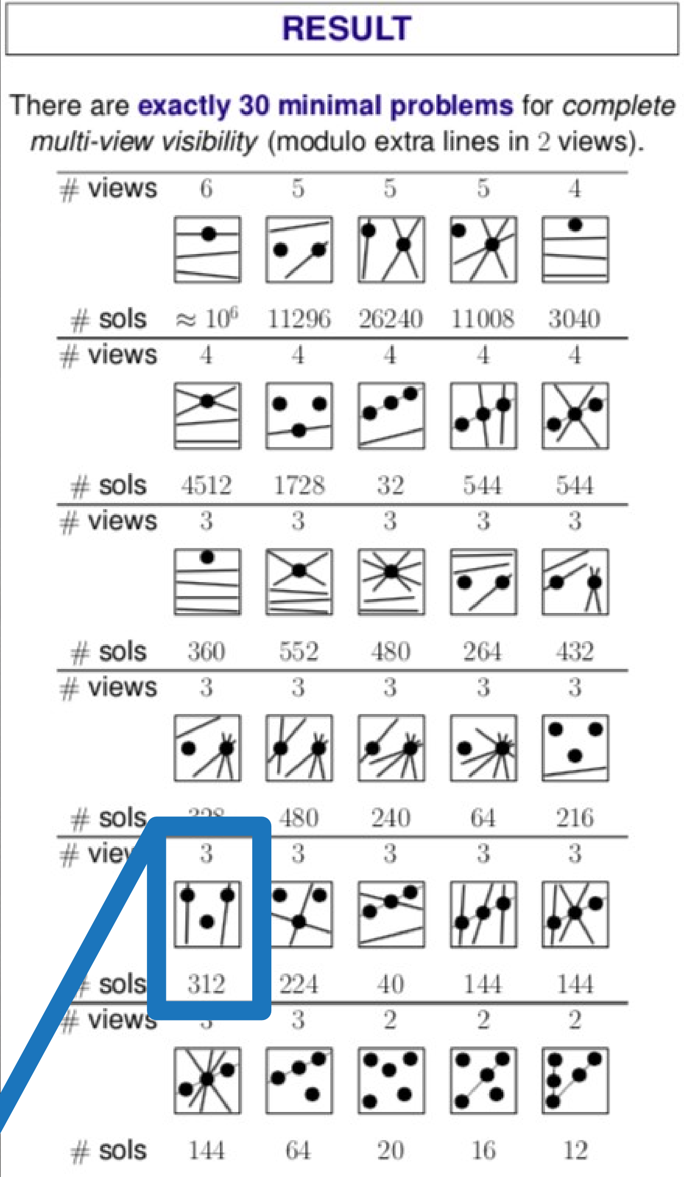
We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image.

First solver for such a high-degree problem based on state-of-the-art algorithms from **numerical algebraic geometry:**

TRPLP – Trifocal Relative Pose from Lines at Points, Fabbri et. al., CVPR 2020



This problem has 312 solutions
(counted over the complex numbers).

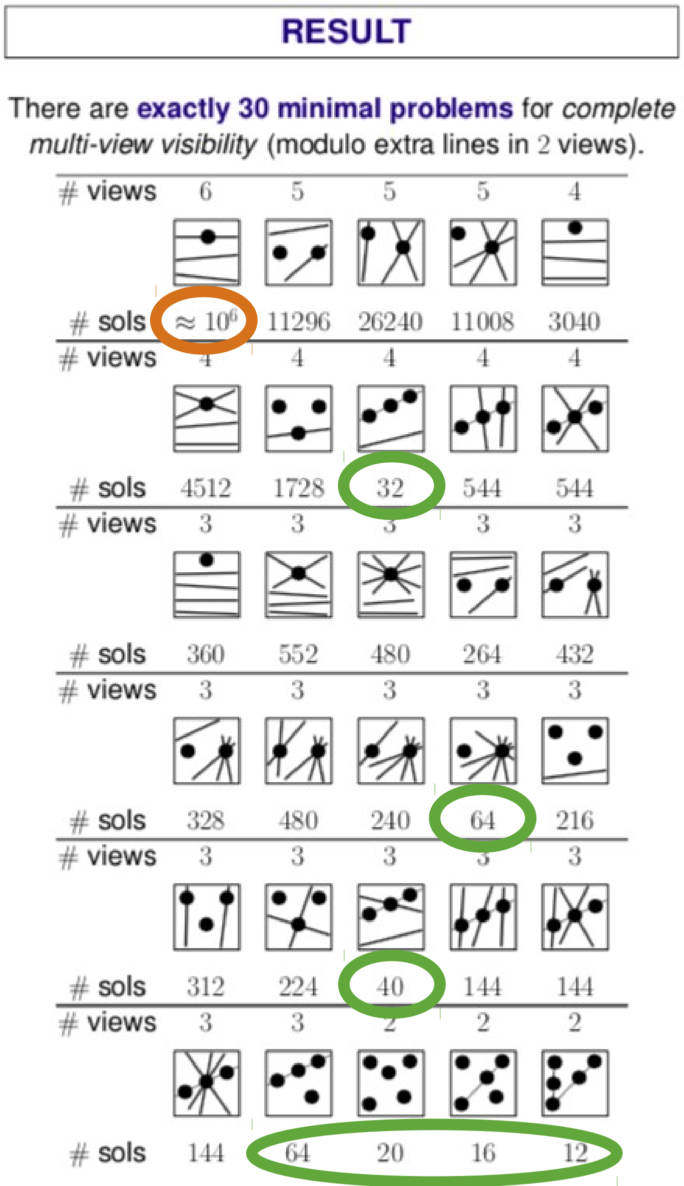


Our Result

We provide the **first complete classification of all minimal problems** when all points and lines are visible in each given image.

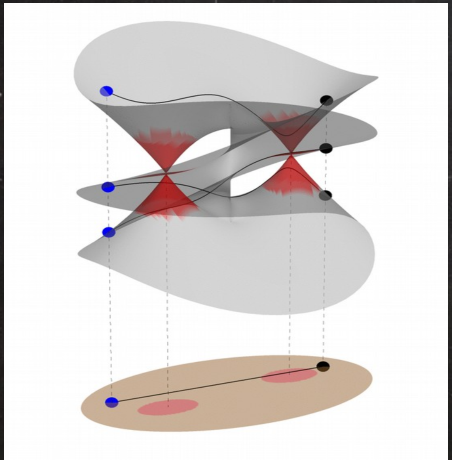
We **measure the complexity of each minimal problem** by computing its number of solutions

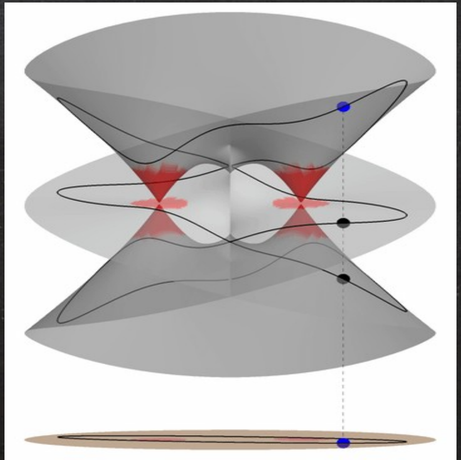
(counted over the complex numbers).

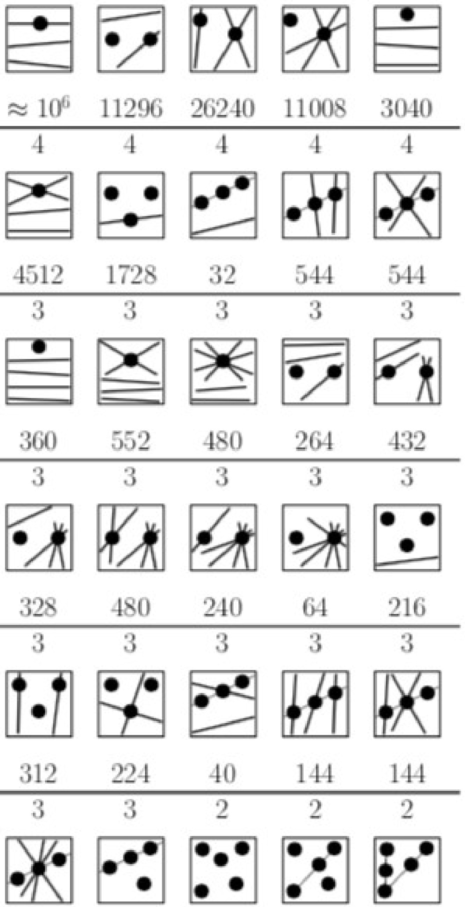


Our Tools: Nonlinear Algebra

- Algebraic geometry
  for proof of classification
- Gröbner bases
  symbolic computation of #sols
  for 2 & 3 views
- Homotopy continuation & monodromy
  numerical computation of #sols
  for 4, 5 & 6 views







RESULT

There are **exactly 30 minimal problems** for *complete multi-view visibility* (modulo extra lines in 2 views).

# views	6	5	5	5	4
# sols	$\approx 10^6$	11296	26240	11008	3040

# views	4	4	4	4	4
# sols	4512	1728	32	544	544

# views	3	3	3	3	3
# sols	360	552	480	264	432

# views	3	3	3	3	3
# sols	328	480	240	64	216

# views	3	3	3	3	3
# sols	312	224	40	144	144

# views	3	3	2	2	2
# sols	144	64	20	16	12

What about partial visibility?

There can be missing data / occlusions in the given images.

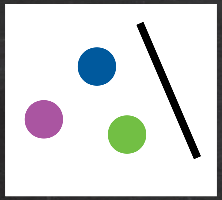


Image 1

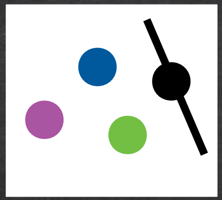


Image 2

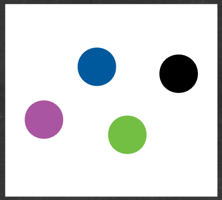


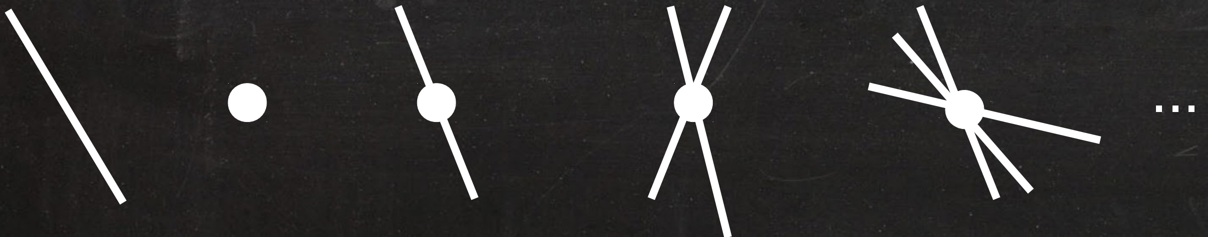
Image 3

- Minimal problems with complete visibility have at most 6 views.
  Minimal problems with partial visibility exists for arbitrarily many views!

  ⟹ Assume: **3 views**

- There are still ∞ minimal problems, and their classification is hard!

  ⟹ Assume: **each line is adjacent to at most 1 point**



There are still ∞ minimal problems!

Our Result

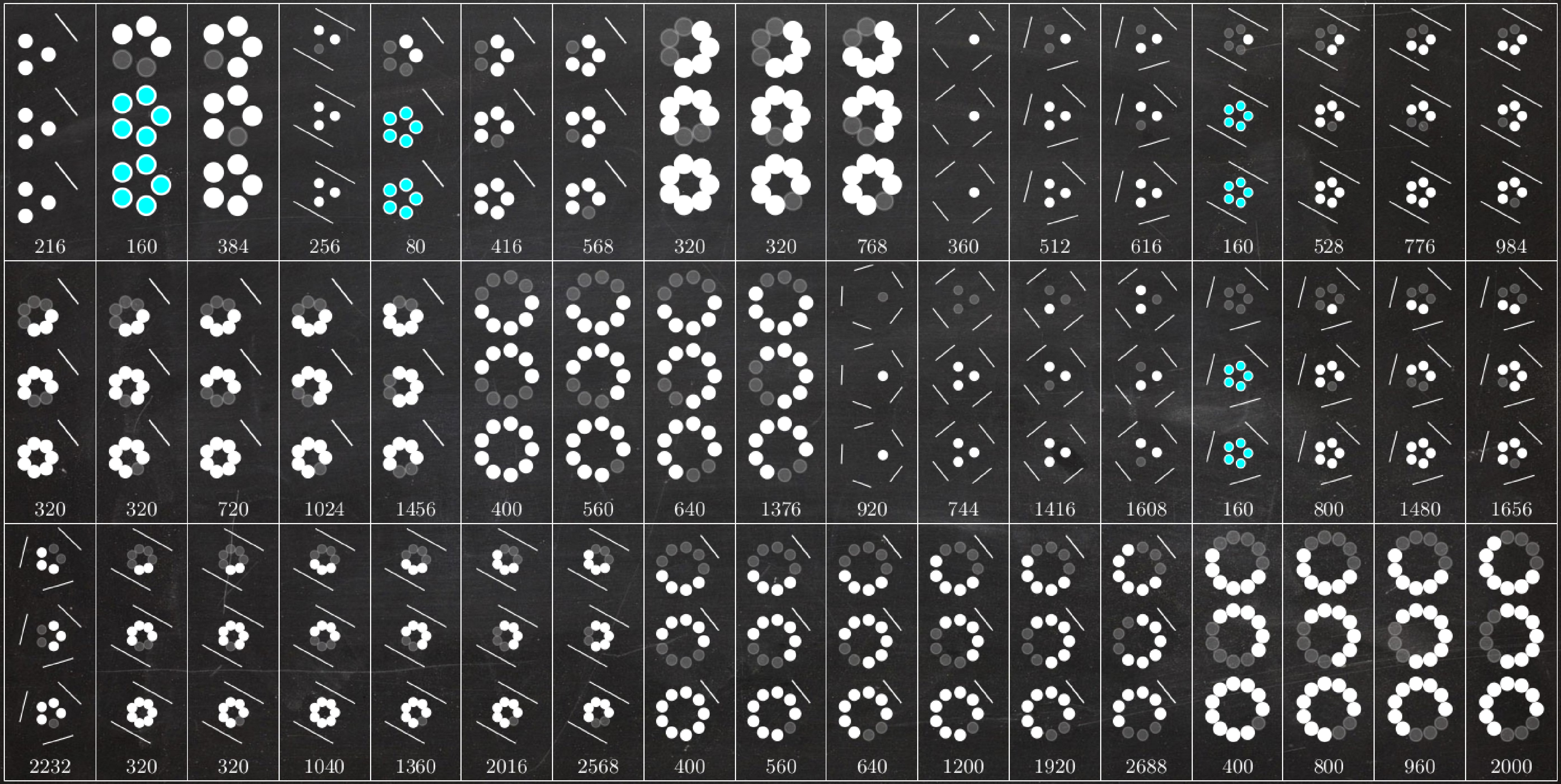
We **completely classify all minimal problems for 3 views** when each line is adjacent to at most 1 point:

There are **74575** equivalence classes of minimal problems.

Among them, **759** have **less than 300** solutions.

# solutions	64	80	144	160	216	224	240	256	264	272	288
# problems	13	9	3	547	7	2	159	2	2	11	4

There are **51** equivalence classes of minimal problems without incidences.



Final comment: Interaction between different sciences is key!

Thanks for your attention!