Moment Varieties of Measures on Polytopes

joint with Boris Shapiro (Stockholms universitet)
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December 14, 2018

Moments of a Probability Distribution

- Let μ be a probability distribution on $\mathbb{R}^d$.
- For $\mathcal{I} = (i_1, \ldots, i_d) \in \mathbb{Z}^d_{\geq 0}$, the $\mathcal{I}$-th **moment** of μ is

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Our goal: Study the dependencies among the moments!

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$$\mathcal{M}_{\mathcal{A}}(\mathcal{F}) := \overline{\operatorname{im} m_{\mathcal{F},\mathcal{A}}} \subset \mathbb{P}^{|\mathcal{A}|-1}$$

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Example: Line Segments

- Let $P = [a, b] \subset \mathbb{R}^1$

$$\begin{aligned}
\Rightarrow \; m_i(P) \; = \; m_i(a,b) \; &= \frac{1}{b-a}\int_a^b w^i \,\mathrm{d}w \; = \; \frac{1}{i+1}\,\frac{b^{i+1}-a^{i+1}}{b-a} \\
&= \frac{1}{i+1}\left(a^i + a^{i-1}b + a^{i-2}b^2 + \ldots + b^i\right)
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$$\Rightarrow \; m_{\diagup,\{0,1,\ldots,r\}} : \mathbb{R}^2 \dashrightarrow \mathbb{P}^r,$$
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- $\mathcal{M}_{\{0,1,\ldots,r\}}(\diagup)$ is a surface in $\mathbb{P}^r$

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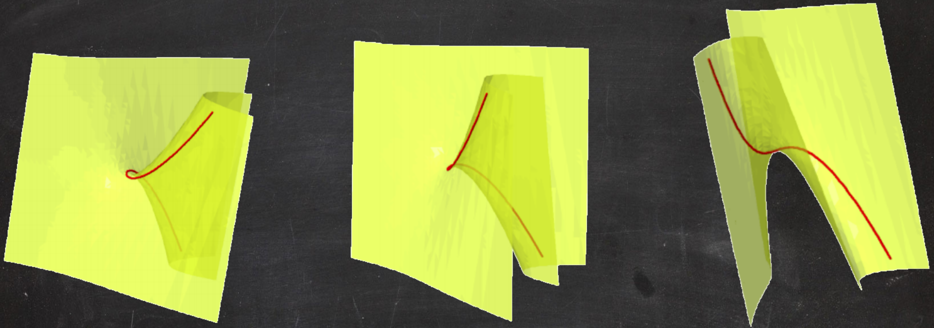

Moment surface $\mathcal{M}_{\{0,1,2,3\}}(\diagup) \subset \mathbb{P}^3$ in affine chart $\{m_0 = 1\}$

- Defined by $2m_1^3 - 3m_0 m_1 m_2 + m_0^2 m_3 = 0$

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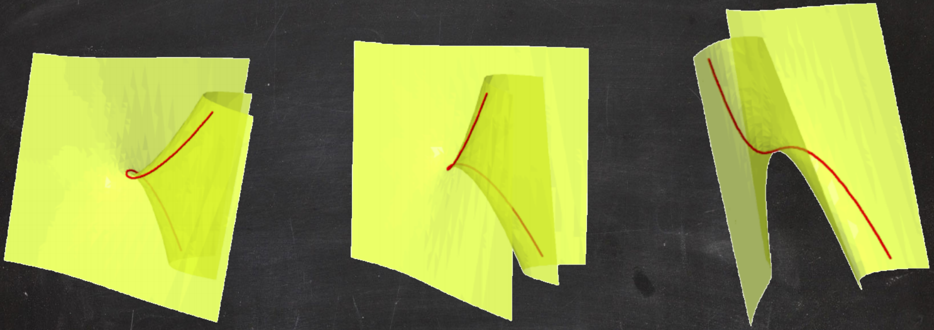

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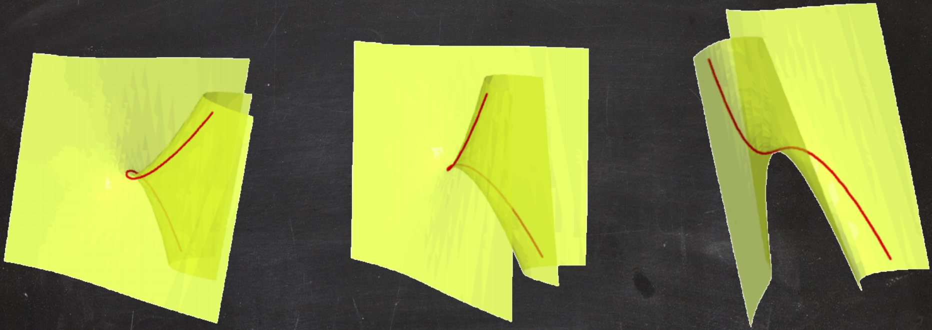

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- Defined by $2m_1^3 - 3m_0 m_1 m_2 + m_0^2 m_3 = 0$
- Singular along $\{m_0 = m_1 = 0\}$
- Contains twisted cubic curve (in red) corresponding to degenerate line segments $[a, a]$ of length 0

Example: Line Segments

The moment surface $\mathcal{M}_{\{0,1,\ldots,r\}}(\diagup) \subset \mathbb{P}^r$

- has degree $\binom{r}{2}$
- and its prime ideal is generated by the 3×3 minors of

$$\begin{pmatrix} 0 & m_0 & 2m_1 & 3m_2 & 4m_3 & \cdots & (r-1)m_{r-2} \\ m_0 & 2m_1 & 3m_2 & 4m_3 & 5m_4 & \cdots & r\,m_{r-1} \\ 2m_1 & 3m_2 & 4m_3 & 5m_4 & 6m_5 & \cdots & (r+1)m_r \end{pmatrix}.$$

- These cubics even form a Gröbner basis.

One-Dimensional Moments

Let $\mathcal{P}$ be any combinatorial type of simplicial polytopes in $\mathbb{R}^d$ with n vertices, and let $\mathcal{A} = \{(0,0,\ldots,0),(1,0,\ldots,0),\ldots,(r,0,\ldots,0)\}$.

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Theorem (K., Shapiro, Sturmfels)
$\mathcal{M}_{\mathcal{A}}(\mathcal{P})$ has degree $\binom{r-n+d+1}{n}$ and its prime ideal is generated by the maximal minors of the Hankel matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_n & c_{n+1} & \cdots & c_{r+d-n} \\ c_1 & c_2 & \cdots & c_{n+1} & c_{n+2} & \cdots & c_{r+d-n+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} & c_{2n+1} & \cdots & c_{r+d} \end{pmatrix},$$

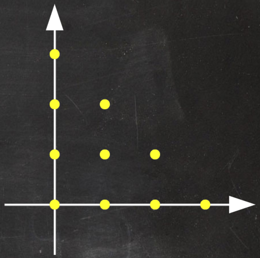
where $c_0 = c_1 = \ldots = c_{d-1} = 0$ and $c_{i+d} = \binom{d+i}{d} m_i$ for $i = 0,1,\ldots,r$.

These minors form a reduced Gröbner basis with respect to any antidiagonal term order, with initial monomial ideal $\langle m_{n-d}, m_{n-d+1}, \ldots, m_{r-n}\rangle^{n+1}$.

Example: Triangles

Let $\mathcal{A}$ be as shown on the right.

The moment variety $\mathcal{M}_{\mathcal{A}}(\triangle) \subset \mathbb{P}^9$ has dimension 6 and degree 30.



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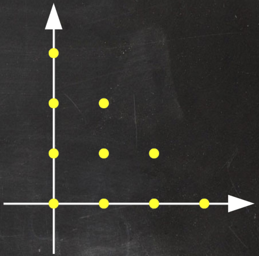

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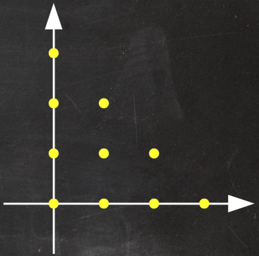

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The $\mathbb{Z}^3$-degrees of the minimal generators of its prime ideal are $(4,2,3), (4,3,2), (4,2,4), (4,3,3), (4,3,3), (4,4,2), (4,3,4), (4,4,3), (6,6,6)$.

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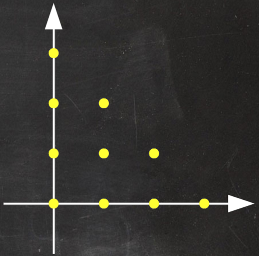

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The ideal generator of degree $(4,2,3)$ equals

$$3m_{02}m_{10}^2m_{01} - 6m_{11}m_{10}m_{01}^2 + 3m_{20}m_{01}^3 - m_{03}m_{10}^2m_{00} + 4m_{11}^2m_{01}m_{00} + m_{21}m_{02}m_{00}^2 \\ - 4m_{20}m_{02}m_{01}m_{00} + 2m_{12}m_{10}m_{01}m_{00} - m_{21}m_{01}^2m_{00} + m_{03}m_{20}m_{00}^2 - 2m_{12}m_{11}m_{00}^2.$$

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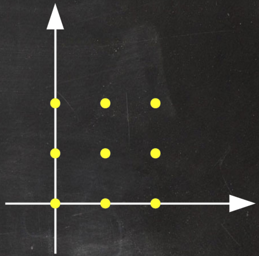

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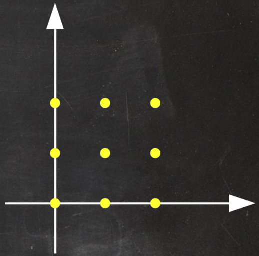

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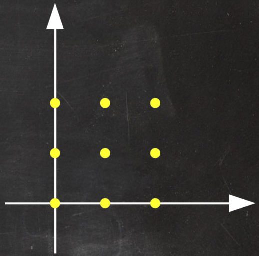

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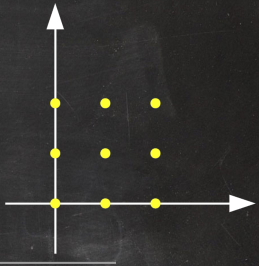

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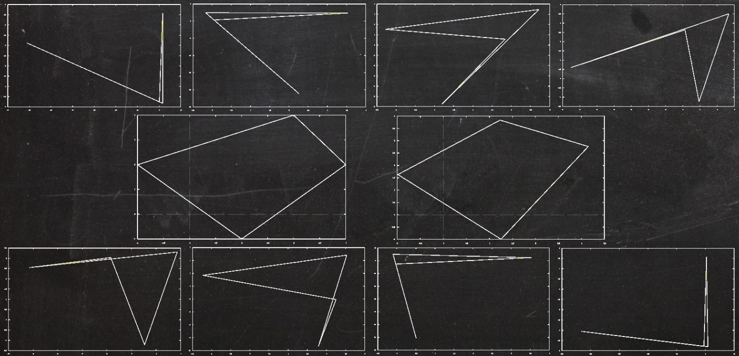

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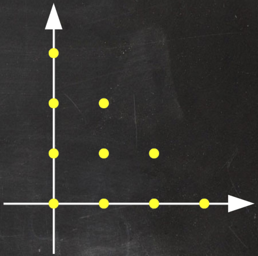

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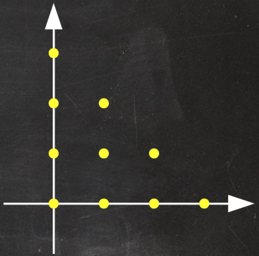

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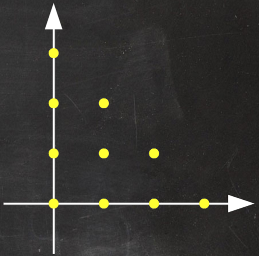

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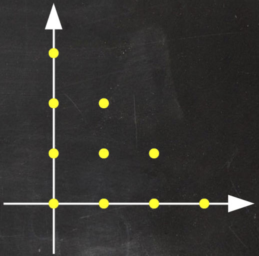

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Lemma:
The defining equation of $\mathcal{M}_{\mathcal{A}}(\square)$ is invariant under the natural action of the affine group Aff_2.

Goal:

- Compute the invariant ring $\mathbb{R}[m_I \mid I \in \mathcal{A}]^{\mathrm{Aff}_2}$
- Express the defining equation of $\mathcal{M}_{\mathcal{A}}(\square)$ in these invariants.

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- The action induces an action on monomials and hence an action on moments:

$$(A, b).m_I = \sum_{J:|J|\leq|I|} \nu_{IJ}(A, b) \cdot m_J,$$

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Example ($d = 1$):
Aff_1 acts on $\mathbb{R}^1$ via $(a, b).x := ax + b$
It acts on moments via $(a, b).m_i = \sum_{j=0}^{i} \binom{i}{j} a^j b^{i-j} m_j$

The Invariant Ring of the Affine Group

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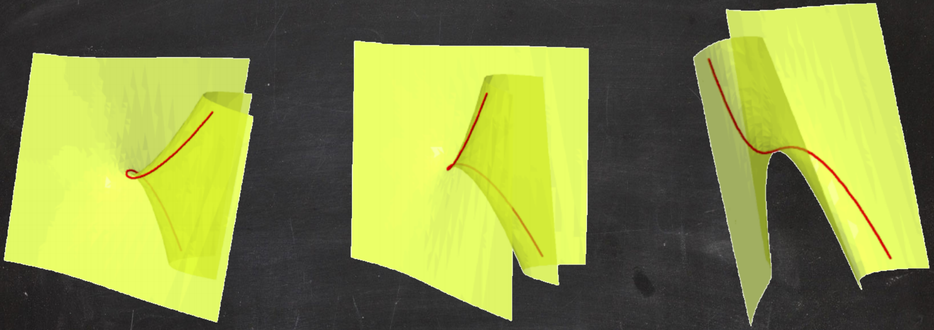
Lemma:
The invariant ring $\mathbb{R}[m_0, m_1, m_2, m_3]^{\mathrm{Aff}_1}$ is generated by

- m_0
- $m_0 m_2 - m_1^2$
- $m_0^2 m_3 - 3 m_0 m_1 m_2 + 2 m_1^3$
- $m_0^2 m_3^2 - 6 m_0 m_1 m_2 m_3 + 4 m_0 m_2^3 + 4 m_1^3 m_3 - 3 m_1^2 m_2^2$

Example: Line Segments



Moment surface $\mathcal{M}_{\{0,1,2,3\}}(\text{LineSegments}) \subset \mathbb{P}^3$ in affine chart $\{m_0 = 1\}$

- Defined by $2m_1^3 - 3m_0 m_1 m_2 + m_0^2 m_3 = 0$
- Singular along $\{m_0 = m_1 = 0\}$
- Contains twisted cubic curve (in red) corresponding to degenerate line segments $[a, a]$ of length 0

The Invariant Ring of the Affine Group

Proposition:
The invariant ring $\mathbb{R}[m_{00}, m_{01}, m_{10}, m_{02}, m_{11}, m_{20}, m_{03}, m_{12}, m_{21}, m_{30}]^{\mathrm{Aff}_2}$ is generated by:

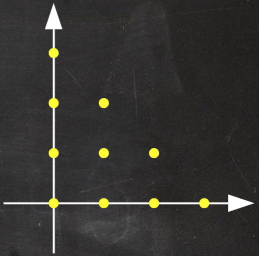
affine invariant	m_{00}	s	t	h	g	j
$\mathbb{Z}^3$-degree	$(1,0,0)$	$(4,4,4)$	$(6,6,6)$	$(3,2,2)$	$(8,6,6)$	$(12,9,9)$
# terms	1	25	103	5	168	892

Back to Quadrilaterals

Let $\mathcal{A} := \{I \in \mathbb{Z}^2_{\geq 0} \mid |I| \leq 3\}$.

The defining equation of the moment hypersurface $\mathcal{M}_{\mathcal{A}}(\square) \subset \mathbb{P}^9$ has $\mathbb{Z}^3$-degree $(\mathbf{18}, 12, 12)$.

It is an Aff_2-invariant.



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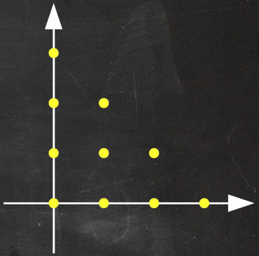

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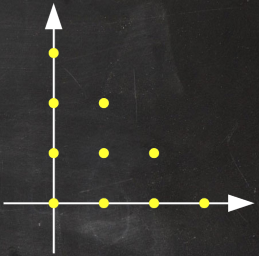

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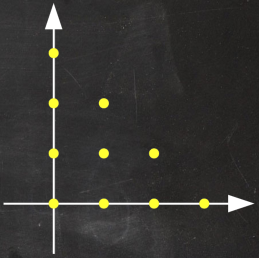

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The hypersurface $\mathcal{M}_{\mathcal{A}}(\square) \subset \mathbb{P}^9$ is defined by

$$2125764\, h^6 + 5484996\, m_{00}^2 h^4 s - 1574640\, m_{00} g h^3 + 364500\, m_{00}^3 h^3 t$$
$$+ 3458700\, m_{00}^4 h^2 s^2 - 2041200 m_{00}^3 ghs + 472500 m_{00}^5 hst - 122500 m_{00}^6 s^3 + 291600 m_{00}^2 g^2$$
$$-135000 m_{00}^4 gt + 15625 m_{00}^6 t^2.$$

This polynomial has 5100 terms in the $m_{i_1 i_2}$.

The moments of order ≤ 3 of probability measures on the triangle $\triangle \subset \mathbb{R}^2$ whose densities are linear functions

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$$\begin{gathered}
12288754756878336m^{16}s^9 - 125913170530271232h^2m^{14}s^8 - 11555266180939776hm^{15}s^7t - 423695444226048m^{16}s^6t^2 \\
-242587475329941504h^4m^{12}s^7 - 678881794908487680h^3m^{13}s^6t - 2253544388296704h^2m^{14}s^5t^2 + 92156256976896hm^{15}s^4t^3 \\
+4239929831616m^{16}s^3t^4 - 2425179321925632ghm^{13}s^7 + 767341894828032gm^{14}s^6t - 1302706722212675584h^6m^{10}s^6 \\
-108262506929061888h^5m^{11}s^5t + 673312350928896h^4m^{12}s^4t^2 + 535497484271616h^3m^{13}s^3t^3 + 31959518257152h^2m^{14}s^2t^4 \\
+440798423040hm^{15}st^5 + 1959367988855543936gh^3m^{11}s^6 - 410140620619776gh^2m^{12}s^5t - 412398826108747776gh^6m^8s^3t \\
-2360537593675776ghm^{13}s^4t^2 - 89805332054016gm^{14}s^3t^3 - 486870353365172224h^8m^8s^5 + 6819936693387264h^7m^9s^4t \\
+29422733985054720h^6m^{10}s^3t^2 + 2782917213290496h^5m^{11}s^2t^3 + 58246341746688h^4m^{12}st^4 - 587731230720h^3m^{13}t^5 \\
+3602104581095424g^2m^{12}s^6 - 1577469804816629760gh^5m^9s^5 - 79828890012352512gh^4m^{10}s^4t - 10700934975848448gh^3m^{11}s^3t^2 \\
-668738492301312gh^2m^{12}s^2t^3 - 10448555212800ghm^{13}st^4 + 275499014400gm^{14}t^5 + 1321196639636946944h^{10}m^6s^4 \\
+814698134331457536h^9m^7s^3t + 92179893357379584h^8m^8s^2t^2 + 2541749079638016h^7m^9st^3 - 13792092880896h^6m^{10}t^4 \\
+5867865494677094 4g^2h^2m^{10}s^5 + 16167862146170880g^2hm^{11}s^4t + 705486447968256g^2m^{12}s^3t^2 - 1103687847816200192gh^7m^7s^4 \\
+13931406950400gh^3m^{11}t^4 - 44584171418419200gh^5m^9s^2t^2 - 9685512225m^{16}t^6 - 1132386035171328gh^4m^{10}st^3 \\
+7839053087502237696h^{12}m^4s^3 + 1352219532013338624h^{11}m^5s^2t + 51427969540816896h^{10}m^6st^2 - 147941222252544h^9m^7t^3 \\
+356552602772570112g^2h^4m^8s^4 + 65355404946702336g^2h^3m^9s^3t + 5201278745444352g^2h^2m^{10}s^2t^2 + 99067782758400g^2hm^{11}st^3 \\
-3265173504000g^2m^{12}t^4 - 5301992678571900928gh^9m^5s^3 - 984505782412247040gh^8m^6s^2t - 37440870596739072gh^7m^7st^2 \\
+260713381625856gh^6m^8t^3 + 7163309458867617792h^{14}m^2s^2 + 495888540219998208h^{13}m^3st - 613682107121664h^{12}m^4t^2 \\
-33414364526542848g^3hm^9s^4 - 2441030167166976g^3m^{10}s^3t + 1297818789047435264g^2h^6m^6s^3 + 235088951956733952g^2h^5m^7s^2t \\
+8250658482290688g^2h^4m^8st^2 - 132090377011200g^2h^3m^9t^3 - 7123133303988682752gh^{11}m^3s^2 - 506754841838616576gh^{10}m^4st \\
+2079004689432576gh^9m^5t^2 + 1846757322198614016h^{16}s - 126388861612851200g^3h^3m^7s^3 - 17847573389770752g^3h^2m^8s^2t \\
-469654673817600g^3hm^9st^2 + 20639121408000g^3m^{10}t^3 + 2594242435278176256g^2h^8m^4s^2 + 183620365983940608g^2h^7m^5st \\
-1848091141472256g^2h^6m^6t^2 - 2445243491429646336gh^{13}ms + 5610807836540928gh^{12}m^2t + 3143555283419136g^4m^8s^3 \\
-408993036765233152g^3h^5m^5s^2 - 26702361435045888g^3h^4m^6st + 626206231756800g^3h^3m^7t^2 + 1246806603479384064g^2h^{10}m^2s \\
-9737274975584256g^2h^9m^3t + 22822562857746432g^4h^2m^6s^2 + 1113255523123200g^4hm^7st - 73383542784000g^4m^8t^2 \\
-299841218941026304g^3h^7m^3s + 5822326385934336g^3h^6m^4t - 12824703626379264g^2h^{12} + 32389413531025408g^4h^4m^4s \\
-1484340697497600g^4h^3m^5t + 15199648742375424g^3h^9m - 1055531162664960g^5hm^5s + 139156940390400g^5m^6t \\
-6878544743366656g^4h^6m^2 + 1407374883553280g^5h^3m^3 - 109951162777600g^6m^4.
\end{gathered}$$

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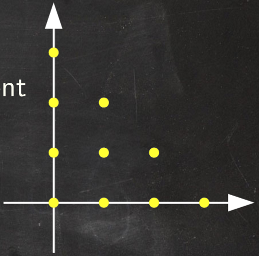
Every partition λ of 10 could possibly yield a moment hypersurface $\mathcal{M}_\lambda(\square) \subset \mathbb{P}^9$.

On the right: $\lambda = 4\ 3\ 2\ 1$



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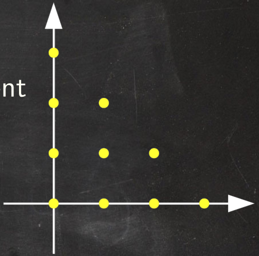

These partitions do not yield hypersurfaces:

λ	λ^c	$\dim \mathcal{M}_\lambda(\Box)$
10	1^{10}	5
9 1	$2\,1^8$	6
8 2	$2^2\,1^6$	7
$8\,1^2$	$3\,1^7$	7

Hypersurfaces $\mathcal{M}_\lambda(\square) \subset \mathbb{P}^9$

λ	λ^c	$\deg \mathcal{M}_\lambda(\square)$
$7\,3$	$2^3\,1^4$	$(5, 10, 0)$
$7\,2\,1$	$3\,2\,1^5$	$(5, 10, 0)$
$7\,1^2$	$4\,1^6$	$(5, 10, 0)$
$6\,4$	$2^4\,1^2$	$(27, 3, 36)$
$6\,3\,1$	$3\,2^2\,1^3$	$(51, 6, 54)$
$6\,2^2$	$3^2\,1^4$	$(96, 12, 90)$
$6\,2\,1^2$	$4\,2\,1^4$	$(136, 18, 126)$
$6\,1^4$	$5\,1^5$	$(480, 72, 424)$
5^2	2^5	$(33, 6, 39)$
$5\,4\,1$	$3\,2^3\,1$	$(36, 6, 36)$
$5\,3\,2$	$3^2\,2\,1^2$	$(42, 12, 36)$
$5\,3\,1^2$	$4\,2^2\,1^2$	$(60, 18, 48)$
$5\,2^2\,1$	$4\,3\,1^3$	$(72, 36, 42)$
$5\,2\,1^3$	$5\,2\,1^3$	$(139, 70, 72)$
$4^2\,2$	$3^2\,2^2$	$(42, 16, 32)$
$4^2\,1^2$	$4\,2^3$	$(60, 24, 42)$
$4\,3^2$	$3^3\,1$	$(47, 20, 34)$
$4\,3\,2\,1$	$4\,3\,2\,1$	$(18, 12, 12)$

Generating Functions

Let $\triangle_d \subset \mathbb{R}^d$ be the d-dimensional simplex.
We denote its vertices by $x_k = (x_{k1}, x_{k2}, \dots, x_{kd})$ for $k = 1, 2, \dots, d+1$.

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Example ($d = 1$): $\triangle_1 = [a, b] \subset \mathbb{R}^1$

$$\sum_{i=0}^{\infty} (i+1) \cdot m_i \cdot t^i = \frac{1}{(1-at)(1-bt)}$$

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Let $P \subset \mathbb{R}^d$ be a simplicial polytope with vertices $x_1, x_2, \ldots, x_n$.
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Thanks for your attention