Point-Line Minimal Problems for 3 Cameras with Partial Visibility

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joint work with Timothy Duff (Georgia Tech),
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Reconstruct 3D scenes and camera poses from 2D images

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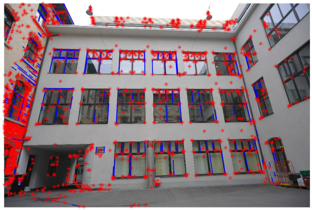
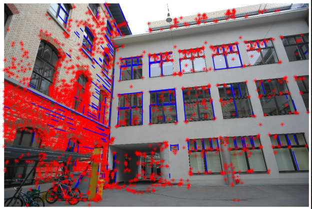
- Step 1: Identify common points and lines on given images





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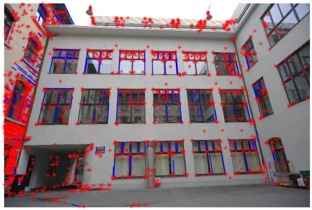 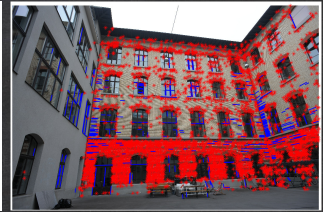 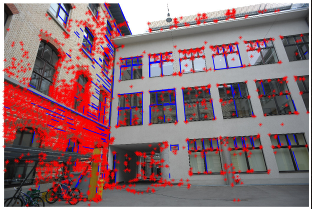

- Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses

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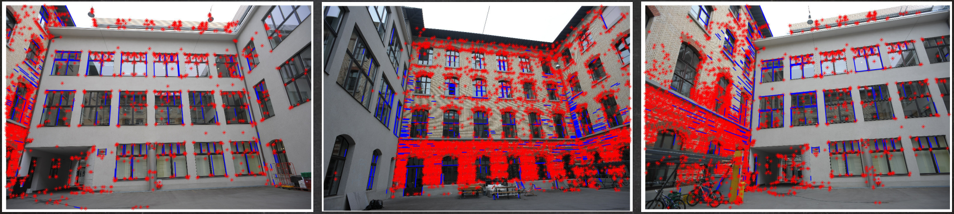


- **Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses**

Reconstruct 3D scenes and camera poses from 2D images

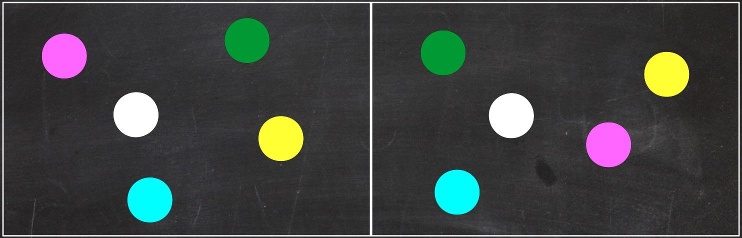
- Step 1: Identify common points and lines on given images

- **Step 2: Reconstruct coordinates of 3D points and lines as well as camera poses**

We use calibrated perspective cameras:
each such camera is represented by a matrix
$[R \mid t]$, where $R \in \mathrm{SO}(3)$ and $t \in \mathbb{R}^3$

5-Point-Problem

Given 2 images of 5 points, recover 5 points in 3D and both camera poses.



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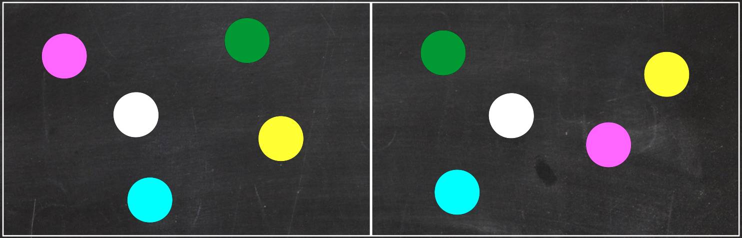

This problem has 20 solutions over $\mathbb{C}$.
(Given 2 images, a solution is 5 points in 3D and 2 camera poses.)

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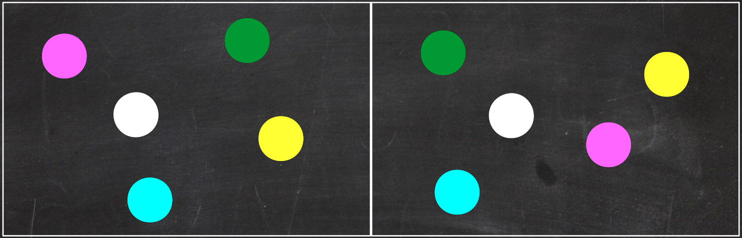

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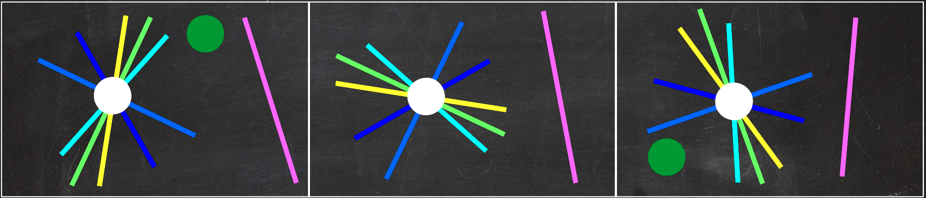
$\Rightarrow$ **The 5-Point-Problem is a minimal problem!**

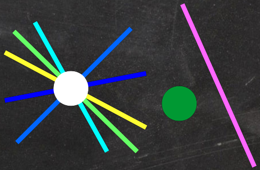
Another minimal problem

with partial visibility

- Given: 3 images like this:



- Recover: 3 camera poses and
  3D coordinates of 2 points and 6 lines
  with the incidences:



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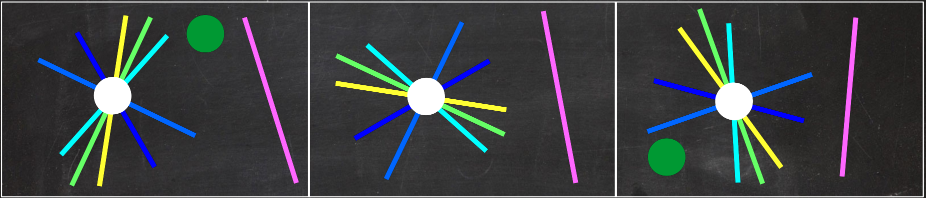

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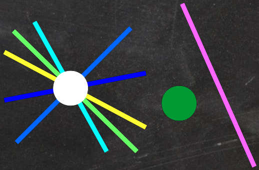

This problem has 240 solutions over $\mathbb{C}$.
(solution = 3 camera poses and 3D coordinates of points and lines)

$\Rightarrow$ **It is a minimal problem!**

Minimal Problems

A **Point-Line-Problem (PLP)** consists of

- a number m of cameras,
- a number p of points,
- a number ℓ of lines,
- a set $\mathcal{I}$ of incidences between points and lines,
- for each camera $c \in \{1, \ldots, m\}$, sets $\mathcal{P}_c$ & $\mathcal{L}_c$ of observed points & lines.

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Definition

A PLP is **minimal** if, given m generic 2D-images, where the c-th image consists of the points and lines in $\mathcal{P}_c$ and $\mathcal{L}_c$ satisfying the incidences $\mathcal{I}$, it has a positive and finite number of solutions over $\mathbb{C}$.

(solution $=$ m camera poses and 3D coordinates of p points and ℓ lines satisfying the incidences $\mathcal{I}$)

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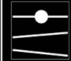
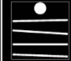
Can we list all minimal PLPs?
How many solutions do they have?

30 Minimal PLPs with Complete Visibility

# views	6	5	5	5	4	4	4	4	4	4
Configuration										
# solutions	$\approx 10^6$	11296	26240	11008	3040	4512	1728	32	544	544
# views	3	3	3	3	3	3	3	3	3	3
Configuration										
# solutions	360	552	480	264	432	328	480	240	64	216
# views	3	3	3	3	3	3	3	2	2	2
Configuration										
# solutions	312	224	40	144	144	144	64	20	16	12

What about Partial Visibility?

1. Minimal PLPs with complete visibility have at most 6 cameras.

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1. Minimal PLPs with complete visibility have at most 6 cameras. **Minimal PLPs with partial visibility exist for arbitrarily many cameras!**
2. Even for a fixed number of cameras, minimal PLPs with partial visibility are much **harder to classify** than those with complete visibility!

Assumptions

1. $m = 3$ cameras
2.
3.
4.

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3. Each intersection point of 2 lines is observed by all cameras which see both lines.
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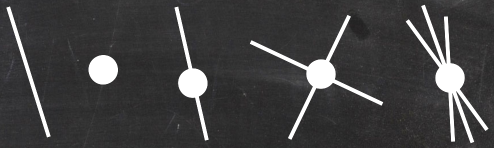
4. Each line is adjacent to at most one point.

ingredients / local features:



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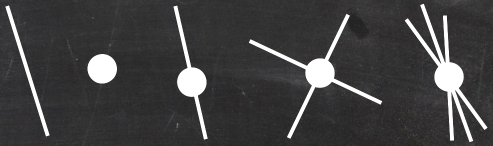

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We call a PLP satisfying these assumptions a **PL1P in 3 views**.

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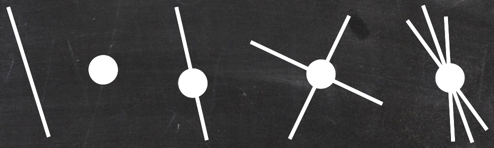

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There are infinitely many minimal PL1Ps in 3 views!!

Reduced PL1Ps

From a PL1P Π one can obtain a new PL1P Π' by forgetting some points and lines (both in 3D-space and in the camera views).

$$\begin{tikzcd}
(\text{3D-arrangement}, \text{cam}_1, \text{cam}_2, \text{cam}_3) \arrow[r, mapsto, "\text{take pictures}"] \arrow[d, mapsto, "\text{forget}"'] & (\text{pic}_1, \text{pic}_2, \text{pic}_3) \arrow[d, mapsto, "\text{forget}"] \\
(\text{3D-arrangement}', \text{cam}_1, \text{cam}_2, \text{cam}_3) \arrow[r, mapsto, "\text{take pictures}"'] & (\text{pic}_1', \text{pic}_2', \text{pic}_3')
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We say that Π is **reducible** to Π' if

1. both are minimal,
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3.

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\text{forget} \downarrow & & \downarrow \text{forget} \\
(\text{3D-arrangement}', \text{cam}_1, \text{cam}_2, \text{cam}_3) & \xmapsto[\text{take pictures}]{} & (\text{pic}'_1, \text{pic}'_2, \text{pic}'_3)
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Definition

We say that Π is **reducible** to Π′ if

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2. for each forgotten point, at most one of its pins is kept, and
3. for generic pictures $(\text{pic}_1, \text{pic}_2, \text{pic}_3)$,
   a generic solution of Π′ on input $(\text{pic}'_1, \text{pic}'_2, \text{pic}'_3)$
   can be lifted to a solution of Π on input $(\text{pic}_1, \text{pic}_2, \text{pic}_3)$.

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Proposition
If a PL1P is reducible to another PL1P,
then both have the same number of solutions (over $\mathbb{C}$).

Counting Reduced Minimal PL1Ps in 3 views

How do they look?

Theorem

A reduced minimal PL1P in 3 views has ≤ 1 point with ≥ 3 pins.

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If such a point exists,

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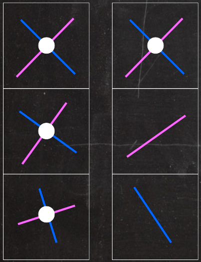
All other local features are viewed as follows:



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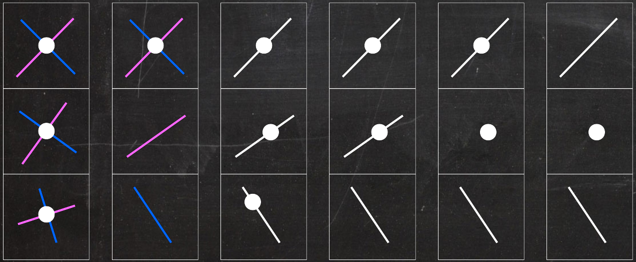

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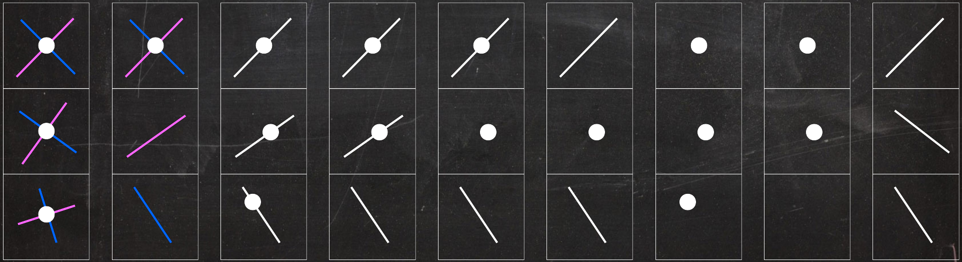

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Degrees of Freedom

3D **2D**

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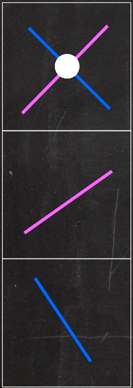
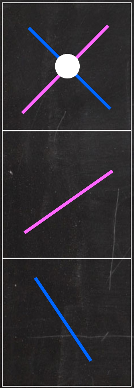


$3 + 2 + 2$

$= 7$

Counting Reduced Minimal PL1Ps in 3 views

Degrees of Freedom



3D	2D
	$2 + 1 + 1$
$3 + 2 + 2$	$+ 2$
	$+ 2$
$= 7$	$= 8$

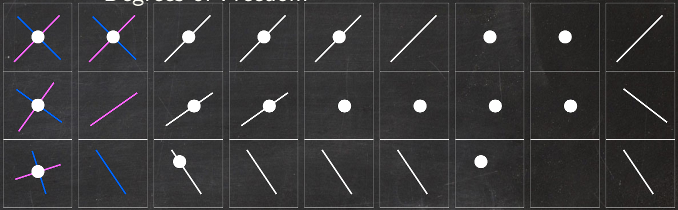
Counting Reduced Minimal PL1Ps in 3 views

Degrees of Freedom

point with

7 6 5 4 3

pins



3D 7

2D 8

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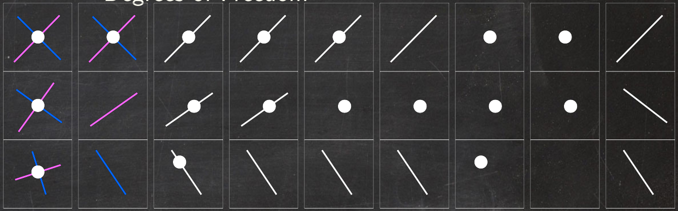

3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6

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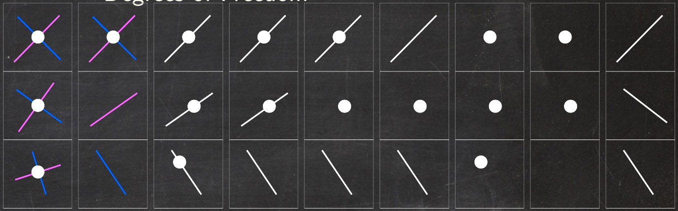

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Lemma: A minimal PL1P in 3 views satisfies:

$$\textbf{degrees of freedom in 3D} + \textbf{camera parameters} = \textbf{degrees of freedom in 2D}$$

Counting Reduced Minimal PL1Ps in 3 views

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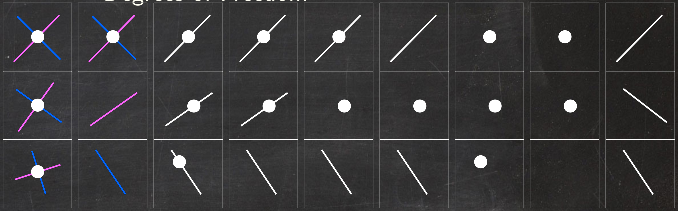

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2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}

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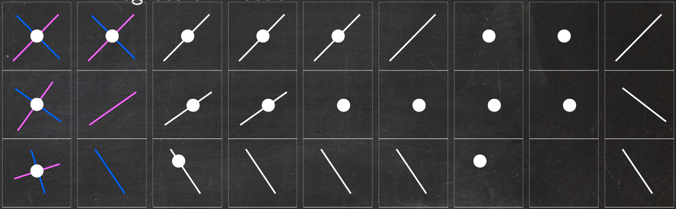

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Lemma: A minimal PL1P in 3 views satisfies:

degrees of freedom in 3D + camera parameters = degrees of freedom in 2D

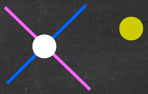
$$17x_1+15x_2+13x_3+11x_4+9x_5+7x_6+7x_7+5x_8+5x_9+5x_{10}+5x_{11}+3x_{12}+3x_{13}+4x_{14} + 11 = 27x_1+24x_2+21x_3+18x_4+15x_5+12x_6+8x_7+9x_8+8x_9+7x_{10}+6x_{11}+6x_{12}+4x_{13}+6x_{14}$$

Permuting single local features...

... in the 3 views changes the PL1P!

Example:

3D



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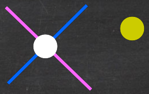

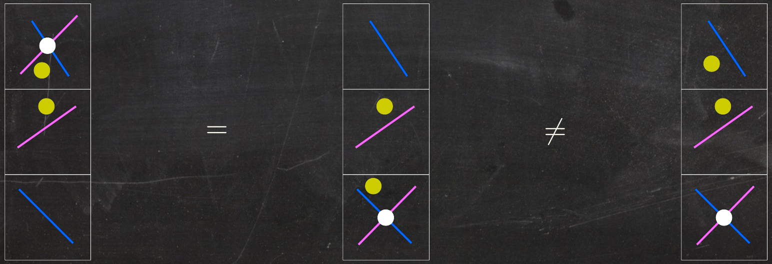

But relabeling the views does not change the PL1P.

Counting Reduced Minimal PL1Ps in 3 views

point with

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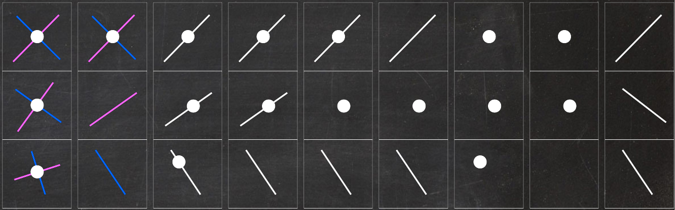

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2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}

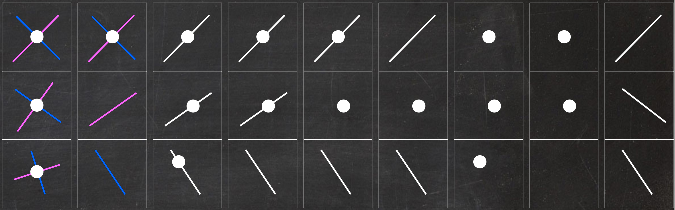
$$\begin{gathered}17x_1+15x_2+13x_3+11x_4+9x_5+7x_6\\+7x_7+5x_8+5x_9+5x_{10}\\+5x_{11}+3x_{12}+3x_{13}+4x_{14}\end{gathered} \quad +11= \quad \begin{gathered}27x_1+24x_2+21x_3+18x_4+15x_5+12x_6\\+8x_7+9x_8+8x_9+7x_{10}\\+6x_{11}+6x_{12}+4x_{13}+6x_{14}\end{gathered}$$

Counting Reduced Minimal PL1Ps in 3 views

point with

7 6 5 4 3

pins



3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
perm	1	1	1	1	1	1		1				1		1

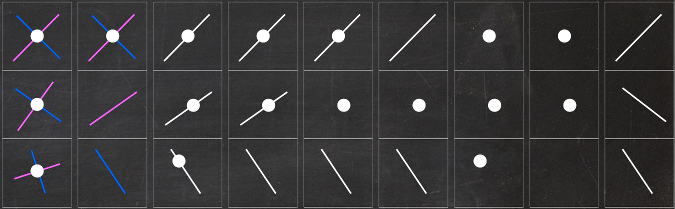
$$17x_1+15x_2+13x_3+11x_4+9x_5+7x_6+7x_7+5x_8+5x_9+5x_{10}+5x_{11}+3x_{12}+3x_{13}+4x_{14}+11=27x_1+24x_2+21x_3+18x_4+15x_5+12x_6+8x_7+9x_8+8x_9+7x_{10}+6x_{11}+6x_{12}+4x_{13}+6x_{14}$$

Counting Reduced Minimal PL1Ps in 3 views

point with

7 6 5 4 3

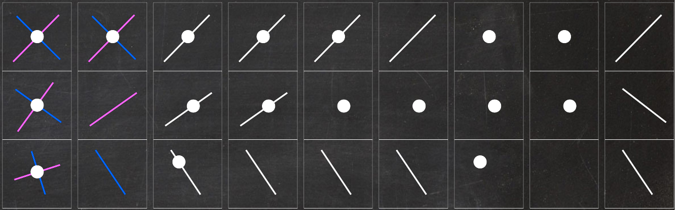
pins



3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
perm	1	1	1	1	1	1	3	1				1	3	1

$$17x_1+15x_2+13x_3+11x_4+9x_5+7x_6+7x_7+5x_8+5x_9+5x_{10}+5x_{11}+3x_{12}+3x_{13}+4x_{14}+11=27x_1+24x_2+21x_3+18x_4+15x_5+12x_6+8x_7+9x_8+8x_9+7x_{10}+6x_{11}+6x_{12}+4x_{13}+6x_{14}$$

Counting Reduced Minimal PL1Ps in 3 views

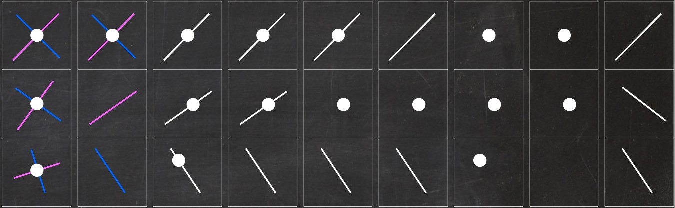


	point with 7 pins	6	5	4	3									
3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7^1, x_7^2, x_7^3	x_8	x_9	x_{10}	x_{11}	x_{12}	$x_{13}^1, x_{13}^2, x_{13}^3$	x_{14}
perm	1	1	1	1	1	1	3	1				1	3	1

$$
\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 && +11= && 27x_1+24x_2+21x_3+18x_4+15x_5+12x_6\\
&+7x_7+5x_8+5x_9+5x_{10} && && +8x_7+9x_8+8x_9+7x_{10}\\
&+5x_{11}+3x_{12}+3x_{13}+4x_{14} && && +6x_{11}+6x_{12}+4x_{13}+6x_{14}
\end{aligned}
$$

Counting Reduced Minimal PL1Ps in 3 views

point with 7 6 5 4 3 pins

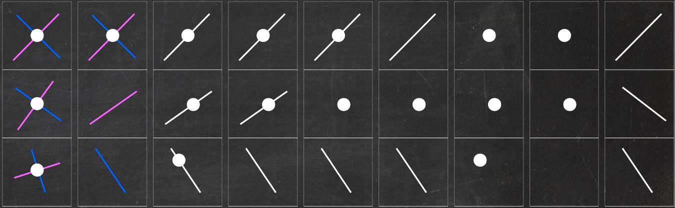


	7	6	5	4	3									
3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7^1, x_7^2, x_7^3	x_8	x_9^1, x_9^2, x_9^3	$x_{10}^1 .. x_{10}^6$	$x_{11}^1, x_{11}^2, x_{11}^3$	x_{12}	$x_{13}^1, x_{13}^2, x_{13}^3$	x_{14}
perm	1	1	1	1	1	1	3	1	3	6	3	1	3	1

$$17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 +7x_7+5x_8+5x_9+5x_{10} +5x_{11}+3x_{12}+3x_{13}+4x_{14} \quad +11= \quad 27x_1+24x_2+21x_3+18x_4+15x_5+12x_6 +8x_7+9x_8+8x_9+7x_{10} +6x_{11}+6x_{12}+4x_{13}+6x_{14}$$

Counting Reduced Minimal PL1Ps in 3 views

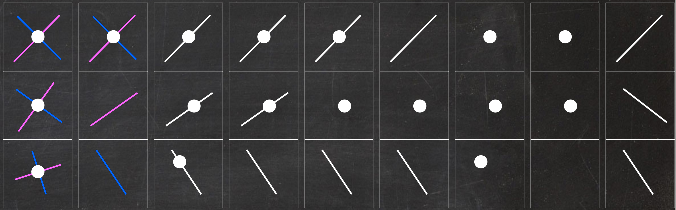
point with 7 6 5 4 3 pins



3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7^1, x_7^2, x_7^3	x_8	x_9^1, x_9^2, x_9^3	$x_{10}^1 .. x_{10}^6$	$x_{11}^1, x_{11}^2, x_{11}^3$	x_{12}	$x_{13}^1, x_{13}^2, x_{13}^3$	x_{14}
perm	1	1	1	1	1	1	3	1	3	6	3	1	3	1

$$
\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 \\
&+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6) \\
&+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14} \\
&+11= \\
&27x_1+24x_2+21x_3+18x_4+15x_5+12x_6 \\
&+8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6) \\
&+6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}
\end{aligned}
$$

Counting Reduced Minimal PL1Ps in 3 views



point with pins	7	6	5	4	3									
3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7^1, x_7^2, x_7^3	x_8	x_9^1, x_9^2, x_9^3	$x_{10}^1 .. x_{10}^6$	$x_{11}^1, x_{11}^2, x_{11}^3$	x_{12}	$x_{13}^1, x_{13}^2, x_{13}^3$	x_{14}
perm	1	1	1	1	1	1	3	1	3	6	3	1	3	1

$$
\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 \\
&+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6) \\
&+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14} \\
&+11= \\
&27x_1+24x_2+21x_3+18x_4+15x_5+12x_6 \\
&+8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6) \\
&+6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}
\end{aligned}
$$

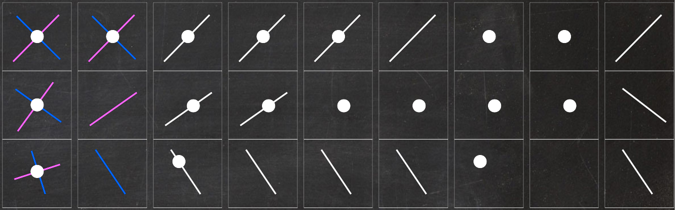
Every reduced minimal PL1Ps in 3 views yields a non-negative integer solution of this equation!

Counting Reduced Minimal PL1Ps in 3 views

point with

7 6 5 4 3

pins



3D	17	15	13	11	9	7	7	5	5	5	5	3	3	4
2D	27	24	21	18	15	12	8	9	8	7	6	6	4	6
#	x_1	x_2	x_3	x_4	x_5	x_6	x_7^1, x_7^2, x_7^3	x_8	x_9^1, x_9^2, x_9^3	$x_{10}^1 .. x_{10}^6$	$x_{11}^1, x_{11}^2, x_{11}^3$	x_{12}	$x_{13}^1, x_{13}^2, x_{13}^3$	x_{14}
perm	1	1	1	1	1	1	3	1	3	6	3	1	3	1

$$17x_1+15x_2+13x_3+11x_4+9x_5+7x_6+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6)+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14}+11=$$

$$27x_1+24x_2+21x_3+18x_4+15x_5+12x_6+8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6)+6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}$$

Every reduced minimal PL1Ps in 3 views yields a non-negative integer solution of this equation!

Which solutions are minimal reduced PL1Ps?

Counting Reduced Minimal PL1Ps in 3 views

Every reduced minimal PL1Ps in 3 views yields a non-negative integer solution of

$$
\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 \\
&+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6) \\
&+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14} \\
&+11= \\
&27x_1+24x_2+21x_3+18x_4+15x_5+12x_6 \\
&+8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6) \\
&+6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}
\end{aligned}
$$

Counting Reduced Minimal PL1Ps in 3 views

Every reduced minimal PL1Ps in 3 views yields a non-negative integer solution of

$$\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 && +11= && 27x_1+24x_2+21x_3+18x_4+15x_5+12x_6\\
&+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6) && && +8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6)\\
&+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14} && && +6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}
\end{aligned}$$

- This equation has **845161** non-negative integer solutions.

Counting Reduced Minimal PL1Ps in 3 views

Every reduced minimal PL1Ps in 3 views yields a non-negative integer solution of

$$\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 && +11= && 27x_1+24x_2+21x_3+18x_4+15x_5+12x_6\\
&+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6) && && +8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6)\\
&+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14} && && +6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}
\end{aligned}$$

- This equation has **845161** non-negative integer solutions.
- Some solutions correspond to PL1Ps which are the same up to relabeling the 3 views.

Counting Reduced Minimal PL1Ps in 3 views

Every reduced minimal PL1Ps in 3 views yields a non-negative integer solution of

$$
\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 \\
&+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6) \\
&+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14} \\
&+11= \\
&27x_1+24x_2+21x_3+18x_4+15x_5+12x_6 \\
&+8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6) \\
&+6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}
\end{aligned}
$$

- This equation has **845161** non-negative integer solutions.
- Some solutions correspond to PL1Ps which are the same up to relabeling the 3 views.
- So the 845161 solutions describe only **143494** different PL1Ps.

Counting Reduced Minimal PL1Ps in 3 views

Every reduced minimal PL1Ps in 3 views yields a non-negative integer solution of

$$
\begin{aligned}
&17x_1+15x_2+13x_3+11x_4+9x_5+7x_6 \\
&+7(x_7^1+x_7^2+x_7^3)+5x_8+5(x_9^1+x_9^2+x_9^3)+5(x_{10}^1+\ldots+x_{10}^6) \\
&+5(x_{11}^1+x_{11}^2+x_{11}^3)+3x_{12}+3(x_{13}^1+x_{13}^2+x_{13}^3)+4x_{14} \\
&+11= \\
&27x_1+24x_2+21x_3+18x_4+15x_5+12x_6 \\
&+8(x_7^1+x_7^2+x_7^3)+9x_8+8(x_9^1+x_9^2+x_9^3)+7(x_{10}^1+\ldots+x_{10}^6) \\
&+6(x_{11}^1+x_{11}^2+x_{11}^3)+6x_{12}+4(x_{13}^1+x_{13}^2+x_{13}^3)+6x_{14}
\end{aligned}
$$

- This equation has **845161** non-negative integer solutions.
- Some solutions correspond to PL1Ps which are the same up to relabeling the 3 views.
- So the 845161 solutions describe only **143494** different PL1Ps.
- **Which of these 143494 PL1Ps are minimal?**

Minimality Check

Lemma

A PL1P in 3 views satisfying the integer equation on the previous slide is **minimal** if and only if the **differential** of the map

$$(\text{3D-arrangement}, \text{cam}_1, \text{cam}_2, \text{cam}_3) \xmapsto{\text{take pictures}} (\text{pic}_1, \text{pic}_2, \text{pic}_3)$$

is **surjective** at a generic point in its domain.

Minimality Check

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A PL1P in 3 views satisfying the integer equation on the previous slide is **minimal** if and only if the **differential** of the map

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It turns out that only **5707** of the 143494 PL1Ps described by the integer equation are **not minimal**.

Minimality Check

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A PL1P in 3 views satisfying the integer equation on the previous slide is **minimal** if and only if the **differential** of the map

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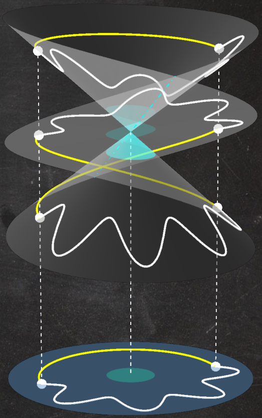
is **surjective** at a generic point in its domain.

It turns out that only **5707** of the 143494 PL1Ps described by the integer equation are **not minimal**.

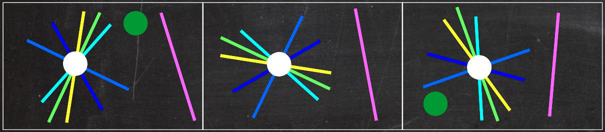
Final Result
There are 137787 reduced minimal PL1Ps in 3 views.

Computing the generic number of solutions

Ongoing work
using homotopy continuation and monodromy
(state-of-the-art methods in
numerical algebraic geometry)



Problem 20 in our list of 137787 minimal problems



has generically **240 solutions**