Chow and Hurwitz Complexes and their Singular Loci

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Introduction to the Grassmannian

\( \text{Gr}(1, \mathbb{P}^3) \)

\( \mathbb{P} := \mathbb{P}_\mathbb{C}, \text{Gr}(1, \mathbb{P}^3) := \{ \text{lines in } \mathbb{P}^3 \} \)

Let \( L \in \text{Gr}(1, \mathbb{P}^3) \) be spanned by rows of \( \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix} \)

\( \Rightarrow \) For \( i < j \), let \( p_{ij} \) be minor of \( \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix} \) using columns \( i, j \)

\( \Rightarrow p_{ij} \)'s are called \text{Plücker coordinates} and satisfy the \text{Plücker relation} \( p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0 \)

This gives the \text{Plücker embedding}

\[ \text{Gr}(1, \mathbb{P}^3) \rightarrow \mathbb{P}^5, \]

\[ L \mapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}). \]

\( \text{Gr}(1, \mathbb{P}^3) \) is 4-dimensional variety in \( \mathbb{P}^5 \) defined by \text{Plücker relation}. 
Section 1

Chow Complexes
Chow complex

Definition
Let \( C \subset \mathbb{P}^3 \) be an irreducible curve.

\[
\text{CH}_0(C) := \left\{ L \in \text{Gr}(1, \mathbb{P}^3) \mid L \cap C \neq \emptyset \right\}
\]

is irreducible hypersurface (i.e., threefold) in \( \text{Gr}(1, \mathbb{P}^3) \), called Chow complex.

\( \Rightarrow \text{CH}_0(C) \) is defined by a polynomial in Plücker coordinates, which is unique up to the Plücker relation, called Chow form.
Degree of Chow complex

Definition
The degree of a line complex \( X \) (i.e., threefold in \( \text{Gr}(1, \mathbb{P}^3) \)) is

\[
\#\{L \in X \mid p \in L \subset H\},
\]

where \( H \subset \mathbb{P}^3 \) is a general plane and \( p \in H \) is a general point. This is also the degree of the defining polynomial of \( X \).

\[ \Rightarrow \deg \text{CH}_0(C) = \deg C \]
Singular locus of Chow complex

**Theorem (K., Nødland, Tripoli)**

If \( \deg C \geq 2 \), then

\[
\text{Sing}(\text{CH}_0(C)) = \text{Sec}(C) \cup \bigcup_{x \in \text{Sing}(C)} G_x,
\]

where

\[
\text{Sec}(C) := \{ L \mid \#(C \cap L) \geq 2 \} \subset \text{Gr}(1, \mathbb{P}^3)
\]

is the **secant congruence** of \( C \) and

\[
G_x := \{ L \mid x \in L \} \subset \text{Gr}(1, \mathbb{P}^3).
\]
Bidegree of secant congruence

Definition
The bidegree of a congruence $X$ (i.e., surface in $\text{Gr}(1, \mathbb{P}^3)$) is

$$\left(\#\{L \in X \mid p \in L\}, \#\{L \in X \mid L \subset H\}\right),$$

where $p \in \mathbb{P}^3$ is a general point and $H \subset \mathbb{P}^3$ is a general plane.

$\Rightarrow \text{bideg}(G_x) = (1, 0)$
**Bidegree of secant congruence**

**Definition**
The bidegree of a congruence $X$ (i.e., surface in $\text{Gr}(1, \mathbb{P}^3)$) is

$$
\left( \#\{L \in X \mid p \in L\}, \#\{L \in X \mid L \subset H\} \right),
$$

where $p \in \mathbb{P}^3$ is a general point and $H \subset \mathbb{P}^3$ is a general plane.

**Theorem (K., Nødland, Tripoli)**

*If C is not contained in any plane and has only ordinary singularities $x_1, \ldots, x_s$ with multiplicities $r_1, \ldots, r_s$, then*

$$
\text{bideg}(\text{Sec}(C)) = \left( \frac{1}{2} (d - 1)(d - 2) - g - \frac{1}{2} \sum_{i=1}^{s} r_i(r_i - 1), \frac{1}{2} d(d - 1) \right),
$$

where $d := \deg(C)$ and $g$ is the geometric genus of $C$.

*If C is contained in a plane and $d \geq 2$, then*

$$
\text{bideg}(\text{Sec}(C)) = (0, \frac{1}{2} d(d - 1)).
$$
Section 2

Hurwitz Complexes
**Hurwitz complex**

**Definition**
Let $S \subset \mathbb{P}^3$ be an irreducible surface, $\deg(S) \geq 2$.

$$\text{CH}_1(S) := \{L \mid L \text{ tangent to } S \text{ at smooth point}\} \subset \text{Gr}(1, \mathbb{P}^3)$$

is irreducible hypersurface (i.e., threefold) in $\text{Gr}(1, \mathbb{P}^3)$, called Hurwitz complex.

Its defining polynomial is called Hurwitz form.
Definition
The degree of a line complex $X$ (i.e., threefold in $\text{Gr}(1, \mathbb{P}^3)$) is

$$\#\{L \in X \mid p \in L \subset H\},$$

where $H \subset \mathbb{P}^3$ is a general point and $p \in H$ is a general point. This is also the degree of the defining polynomial of $X$.

$\Rightarrow$ For general surface $S$ of degree $d$: $\deg(\text{CH}_1(S)) = d(d - 1)$
Singular locus of Hurwitz complex

**Theorem (K., Nødland, Tripoli)**

If $S$ is smooth, does not contain any lines, and $\deg(S) \geq 4$, then

$$\text{Sing}(\text{CH}_1(S)) = \text{Bit}(S) \cup \text{Infl}(S),$$

where

$$\text{Bit}(S) := \{L | L \text{ tangent to } S \text{ at two smooth points}\} \subset \text{Gr}(1, \mathbb{P}^3)$$

is the **bitangent congruence** of $S$ and

$$\text{Infl}(S) := \{L | L \text{ intersects } S \text{ at some smooth point with multiplicity } 3\} \subset \text{Gr}(1, \mathbb{P}^3)$$

is the **inflectional congruence** of $S$. 
Bidegree of bitangent and inflectional congruence

Definition
The bidegree of a congruence $X$ (i.e., surface in $\text{Gr}(1, \mathbb{P}^3)$) is

$$\left( \# \{ L \in X \mid p \in L \}, \# \{ L \in X \mid L \subset H \} \right),$$

where $p \in \mathbb{P}^3$ is a general point and $H \subset \mathbb{P}^3$ is a general plane.

Theorem (K., Nødland, Tripoli; Arrondo, Bertolini, Turrini)
If $S$ is a general surface of degree $d \geq 4$, then

$$\text{bideg}(\text{Bit}(S)) = \left( \frac{1}{2} d(d - 1)(d - 2)(d - 3), \frac{1}{2} d(d - 2)(d - 3)(d + 3) \right),$$

$$\text{bideg}(\text{Infl}(S)) = (d(d - 1)(d - 2), 3d(d - 2)).$$
Section 3

Duality
Projectively dual varieties

Definition

$(\mathbb{P}^n)^*$ is the projectivization of the dual vector space $(\mathbb{C}^{n+1})^*$. Equivalently, $(\mathbb{P}^n)^* = \{ H \subset \mathbb{P}^n \mid \text{hyperplane}\}$. 

$\Rightarrow (\mathbb{P}^n)^* = \mathbb{P}^n$

Definition

Let $X \subset \mathbb{P}^n$ be an irreducible variety.

$X^\vee := \{ H \mid H \text{ tangent to } X \text{ at smooth point} \} \subset (\mathbb{P}^n)^*$

is an irreducible variety, called the projectively dual variety to $X$. 

(Figure: Gel’fand-Kapranov-Zelevinsky)
Theorem (e.g. Gel’fand-Kapranov-Zelevinsky)

\((X^\vee)^\vee = X\).

For smooth points \(x \in X, H \in X^\vee\):

\(H\) is tangent to \(X\) at \(x\) \(\iff\) \(x\) is tangent to \(X^\vee\) at \(H\).
**Plücker’s formula**

**Proposition (e.g. Griffiths-Harris)**

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d$ with exactly $\kappa$ cusps and $\nu$ nodes as singularities. Then

$$\deg(C^\vee) = d(d - 1) - 3\kappa - 2\nu.$$  

**Corollary**

For general surface $S \subset \mathbb{P}^3$ of degree $d$:

$$\deg(CH_1(S)) = \deg((S \cap H)^\vee) = d(d - 1),$$

where $H \subset \mathbb{P}^3$ is a general plane.
Proof of

\[ \text{bideg}(\text{Bit}(S)) = \frac{1}{2} d(d-1)(d-2)(d-3), \]
\[ \text{bideg}(\text{Infl}(S)) = (d(d-1)(d-2), \]

lines through general point

Corollary

A general irreducible plane curve of degree \( d \) has

\[ \frac{1}{2} d(d-2)(d-3)(d+3) \] bitangents and \( 3d(d-2) \) inflectional lines.
Proof of

\[ \text{bideg}(\text{Bit}(S)) = \left( \frac{1}{2} d(d - 1)(d - 2)(d - 3), \right. \]
\[ \left. (d(d - 1)(d - 2), \right) \]

lines through general point

\[ \text{bideg}(\text{Infl}(S)) = \left( \frac{1}{2} d(d - 2)(d - 3)(d + 3), \right. \]
\[ \left. \frac{1}{3} d(d - 2) \right) \]

lines in general plane

bitangents to \( S \) through \( p \) \( \leftrightarrow \) nodes of \( C \)
inflectional lines to \( S \) through \( p \) \( \leftrightarrow \) cusps of \( C \)
Congruences of dual varieties

Theorem (K., Nødland, Tripoli)
Let $C \subset \mathbb{P}^3$ be an irreducible smooth curve, not contained in any plane. Then

$$L \in \text{CH}_0(C) \iff L^\vee \in \text{CH}_1(C^\vee),$$

$$L \in \text{Sec}(C) \iff L^\vee \in \text{Bit}(C^\vee),$$

$L$ tangent to $C \iff L^\vee \in \text{Infl}(C^\vee)$.

Thanks for your attention!