Chow and Hurwitz Complexes and their Singular Loci

Kathlén Kohn[1]
(joint work with B.I. Nødland[2], P. Tripoli[3])

[1]Institute of Mathematics, TU Berlin
[2]Department of Mathematics, University of Oslo
[3]Mathematics Institute, University of Warwick

December 8, 2016

Introduction to the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$

$\mathbb{P} := \mathbb{P}_{\mathbb{C}}$, $\mathrm{Gr}(1, \mathbb{P}^3) := \{$ lines in $\mathbb{P}^3\}$

Let $L \in \mathrm{Gr}(1, \mathbb{P}^3)$ be spanned by rows of $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$
$\Rightarrow$ For $i < j$, let p_{ij} be minor of $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$ using columns i, j
$\Rightarrow$ p_{ij}'s are called Plücker coordinates and satisfy the Plücker relation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$

This gives the Plücker embedding

$$\begin{aligned} \mathrm{Gr}(1, \mathbb{P}^3) &\longrightarrow \mathbb{P}^5, \\ L &\longmapsto (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}). \end{aligned}$$

$\mathrm{Gr}(1, \mathbb{P}^3)$ is 4-dimensional variety in $\mathbb{P}^5$ defined by Plücker relation.

Section 1

Chow Complexes

Chow complex

Definition

Let $C \subset \mathbb{P}^3$ be an irreducible curve.

$$\mathrm{CH}_0(C) := \left\{ L \in \mathrm{Gr}(1, \mathbb{P}^3) \mid L \cap C \neq \varnothing \right\}$$

is irreducible hypersurface (i.e., threefold) in $\mathrm{Gr}(1, \mathbb{P}^3)$, called Chow complex.

$\Rightarrow \mathrm{CH}_0(C)$ is defined by a polynomial in Plücker coordinates, which is unique up to the Plücker relation, called Chow form.

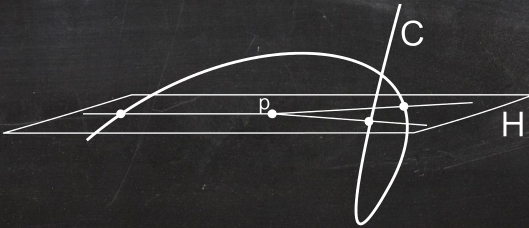
Degree of Chow complex

Definition
The degree of a line complex X (i.e., threefold in $\mathrm{Gr}(1,\mathbb{P}^3)$) is

$$\#\{L \in X \mid p \in L \subset H\},$$

where $H \subset \mathbb{P}^3$ is a general plane and $p \in H$ is a general point. This is also the degree of the defining polynomial of X.

$\Rightarrow \deg \mathrm{CH}_0(C) = \deg C$



Singular locus of Chow complex

Theorem (K., Nødland, Tripoli)
If $\deg C \geq 2$*, then*

$$\mathrm{Sing}(\mathrm{CH}_0(C)) = \mathrm{Sec}(C) \cup \bigcup_{x \in \mathrm{Sing}(C)} \mathcal{G}_x,$$

where

$$\mathrm{Sec}(C) := \overline{\{L \mid \#(C \cap L) \geq 2\}} \subset \mathrm{Gr}(1, \mathbb{P}^3)$$

is the secant congruence of C *and*

$$\mathcal{G}_x := \{L \mid x \in L\} \subset \mathrm{Gr}(1, \mathbb{P}^3)\,.$$

Bidegree of secant congruence

Definition
The bidegree of a congruence X (i.e., surface in $\mathrm{Gr}(1,\mathbb{P}^3)$) is

$$(\#\{L \in X \mid p \in L\},\ \#\{L \in X \mid L \subset H\}),$$

where $p \in \mathbb{P}^3$ is a general point and $H \subset \mathbb{P}^3$ is a general plane.

$\Rightarrow$ $\mathrm{bideg}(\mathcal{G}_X) = (1,0)$

Bidegree of secant congruence

Definition

The bidegree of a congruence X (i.e., surface in $\mathrm{Gr}(1,\mathbb{P}^3)$) is

$$(\#\{L \in X \mid p \in L\},\ \#\{L \in X \mid L \subset H\}),$$

where $p \in \mathbb{P}^3$ is a general point and $H \subset \mathbb{P}^3$ is a general plane.

Theorem (K., Nødland, Tripoli)

If C is not contained in any plane and has only ordinary singularities $x_1, \ldots, x_s$ with multiplicities $r_1, \ldots, r_s$, then

$$\mathrm{bideg}(\mathrm{Sec}(C)) = \left(\frac{1}{2}(d-1)(d-2) - g - \frac{1}{2}\sum_{i=1}^{s} r_i(r_i-1),\ \frac{1}{2}d(d-1)\right),$$

where $d := \deg(C)$ and g is the geometric genus of C.
If C is contained in a plane and $d \geq 2$, then
$\mathrm{bideg}(\mathrm{Sec}(C)) = (0, \frac{1}{2}d(d-1))$.

Section 2

Hurwitz Complexes

Hurwitz complex

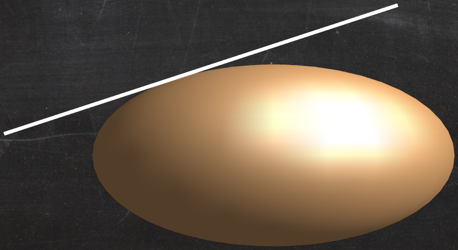
Definition

Let $S \subset \mathbb{P}^3$ be an irreducible surface, $\deg(S) \geq 2$.

$$CH_1(S) := \overline{\{L \mid L \text{ tangent to } S \text{ at smooth point}\}} \subset Gr(1, \mathbb{P}^3)$$

is irreducible hypersurface (i.e., threefold) in $Gr(1, \mathbb{P}^3)$, called Hurwitz complex.

Its defining polynomial is called Hurwitz form.



Degree of Hurwitz complex

Definition
The degree of a line complex X (i.e., threefold in $\mathrm{Gr}(1, \mathbb{P}^3)$) is

$$\#\{L \in X \mid p \in L \subset H\},$$

where $H \subset \mathbb{P}^3$ is a general point and $p \in H$ is a general point.
This is also the degree of the defining polynomial of X.

$\Rightarrow$ For general surface S of degree d: $\deg(\mathrm{CH}_1(S)) = d(d-1)$

Singular locus of Hurwitz complex

Theorem (K., Nødland, Tripoli)
If S is smooth, does not contain any lines, and $\deg(S) \geq 4$, then

$$\mathrm{Sing}(\mathrm{CH}_1(S)) = \mathrm{Bit}(S) \cup \mathrm{Infl}(S),$$

where

$$\mathrm{Bit}(S) := \overline{\{L \mid L \text{ tangent to } S \text{ at two smooth points}\}} \subset \mathrm{Gr}(1, \mathbb{P}^3)$$

is the bitangent congruence of S and

$$\mathrm{Infl}(S) := \overline{\{L \mid L \text{ intersects } S \text{ at some smooth point with multiplicity } 3\}} \subset \mathrm{Gr}(1, \mathbb{P}^3)$$

is the inflectional congruence of S.

Bidegree of bitangent and inflectional congruence

Definition
The bidegree of a congruence X (i.e., surface in $\mathrm{Gr}(1,\mathbb{P}^3)$) is

$$(\#\{L \in X \mid p \in L\},\ \#\{L \in X \mid L \subset H\}),$$

where $p \in \mathbb{P}^3$ is a general point and $H \subset \mathbb{P}^3$ is a general plane.

Theorem (K., Nødland, Tripoli; Arrondo, Bertolini, Turrini)
If S is a general surface of degree $d \geq 4$, then

$$\mathrm{bideg}(\mathrm{Bit}(S)) = \left(\frac{1}{2}d(d-1)(d-2)(d-3),\ \frac{1}{2}d(d-2)(d-3)(d+3)\right),$$
$$\mathrm{bideg}(\mathrm{Infl}(S)) = (d(d-1)(d-2),\ 3d(d-2)).$$

Section 3

Duality

Projectively dual varieties

Definition

$(\mathbb{P}^n)^*$ is the projectivization of the dual vector space $(\mathbb{C}^{n+1})^*$.
Equivalently $(\mathbb{P}^n)^* = \{H \subset \mathbb{P}^n \mid \text{ hyperplane}\}$.

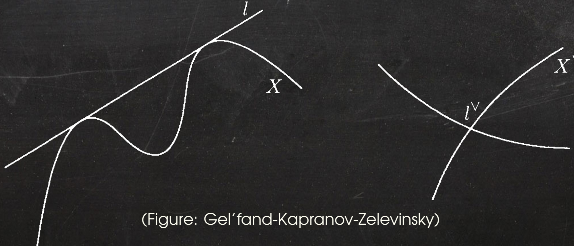
$\Rightarrow \left((\mathbb{P}^n)^*\right)^* = \mathbb{P}^n$

Definition

Let $X \subset \mathbb{P}^n$ be an irreducible variety.

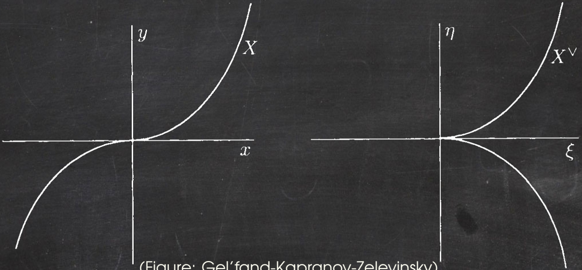
$$X^\vee := \overline{\{H \mid H \text{ tangent to } X \text{ at smooth point}\}} \subset (\mathbb{P}^n)^*$$

is an irreducible variety, called the projectively dual variety to X.



(Figure: Gel'fand-Kapranov-Zelevinsky)

Biduality



(Figure: Gel′fand-Kapranov-Zelevinsky)

Theorem (e.g. Gel'fand-Kapranov-Zelevinsky)

$(X^\vee)^\vee = X.$

For smooth points $x \in X, H \in X^\vee$:

H *is tangent to* X *at* $x \Leftrightarrow x$ *is tangent to* $X^\vee$ *at* H.

Plücker's formula

Proposition (e.g. Griffiths-Harris)

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d with exactly κ cusps and ν nodes as singularities. Then

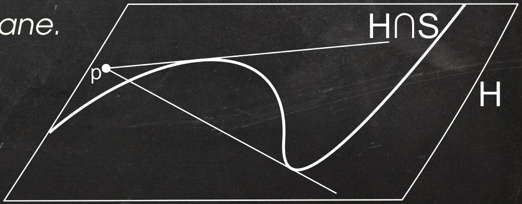
$$\deg(C^\vee) = d(d-1) - 3\kappa - 2\nu.$$

Corollary

For general surface $S \subset \mathbb{P}^3$ of degree d:

$$\deg(\mathrm{CH}_1(S)) = \deg\left((S \cap H)^\vee\right) = d(d-1),$$

where $H \subset \mathbb{P}^3$ is a general plane.

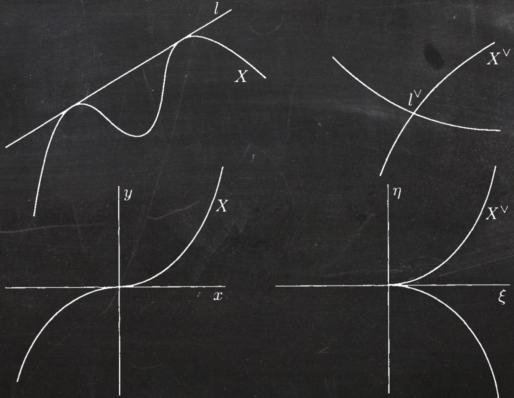


Proof of

bideg(Bit(S)) =	$\big(\frac{1}{2}d(d-1)(d-2)(d-3),$	$\frac{1}{2}d(d-2)(d-3)(d+3)\big)$
bideg(Infl(S)) =	$(d(d-1)(d-2),$	$3d(d-2))$
	lines through general point	lines in general plane

Corollary

A general irreducible plane curve of degree d has $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents and $3d(d-2)$ inflectional lines.

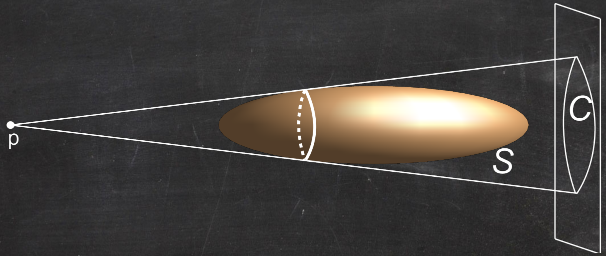


Proof of

$$\mathrm{bideg}(\mathrm{Bit}(S)) = \left(\tfrac{1}{2}d(d-1)(d-2)(d-3),\ \tfrac{1}{2}d(d-2)(d-3)(d+3)\right)$$

$$\mathrm{bideg}(\mathrm{Infl}(S)) = \left(d(d-1)(d-2),\ 3d(d-2)\right)$$

lines through general point lines in general plane



bitangents to S through p $\leftrightsquigarrow$ nodes of C

inflectional lines to S through p $\leftrightsquigarrow$ cusps of C

Congruences of dual varieties

Theorem (K., Nødland, Tripoli)
Let $C \subset \mathbb{P}^3$ be an irreducible smooth curve, not contained in any plane. Then

$$L \in \mathrm{CH}_0(C) \Leftrightarrow L^\vee \in \mathrm{CH}_1(C^\vee),$$
$$L \in \mathrm{Sec}(C) \Leftrightarrow L^\vee \in \mathrm{Bit}(C^\vee),$$
$$L \text{ tangent to } C \Leftrightarrow L^\vee \in \mathrm{Infl}(C^\vee).$$

Thanks for your attention!