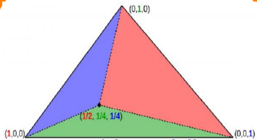
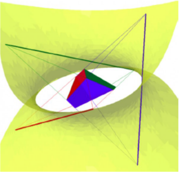
The Adjoint of a Polytope

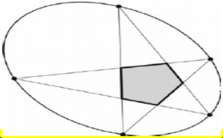
Kathlén Kohn
KTH

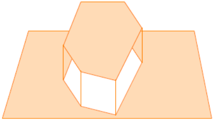


geometric modeling
barycentric coordinates
for arbitrary polytopes

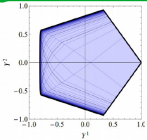


classical algebraic geometry
adjoint hypersurfaces

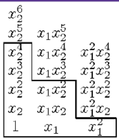




algebraic statistics:
moments of uniform distributions
on polytopes



physics
scattering amplitudes



intersection theory:
Segre classes of
monomial schemes

joint works with Kristian Ranestad (Universitetet i Oslo) /
Boris Shapiro (Stockholms universitet) & Bernd Sturmfels (MPI MiS Leipzig / UC Berkeley)

The Adjoint of a Polygon

Wachspress (1975)

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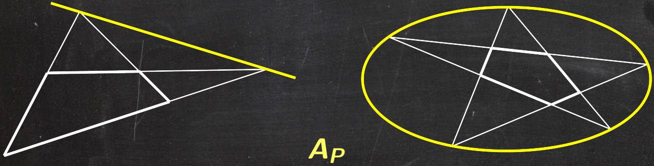
Definition

The **adjoint A_P** of a polygon $P \subset \mathbb{P}^2$ is the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P.



$(\deg A_P = |V(P)| - 3)$

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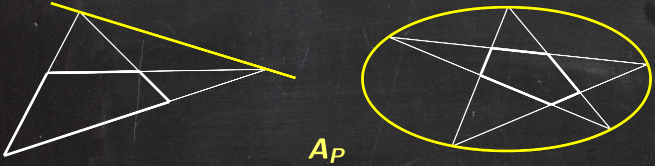

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Generalization to higher-dimensional polytopes?

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Warren (1996)

- P: convex polytope in $\mathbb{R}^n$
- $V(P)$: set of vertices of P
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$$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$.

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$\mathrm{adj}_{\tau(P)}(t)$ *is independent of the triangulation* $\tau(P)$*. So* $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$.

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II *If* P *is a polygon, then* $Z(\mathrm{adj}_P) = A_{P^*}$.
(Recall: $P^* = \{x \in \mathbb{R}^n \mid \forall v \in V(P) : \ell_v(x) \geq 0\}$ *dual polytope of* P*)*

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Geometric definition using a vanishing condition à la Wachspress?

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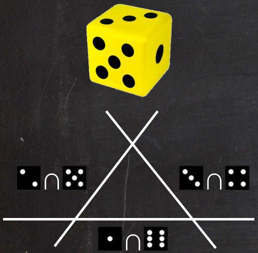

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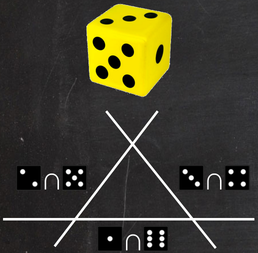

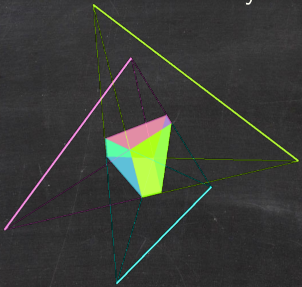

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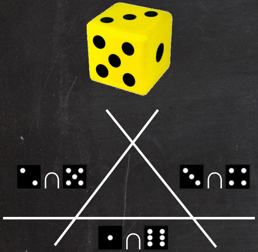

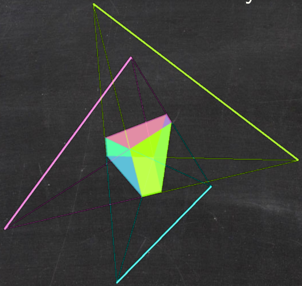

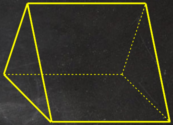

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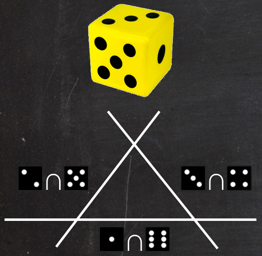

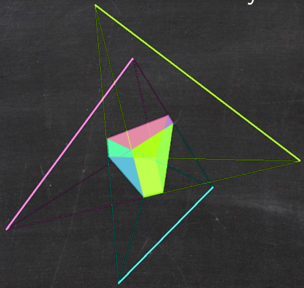

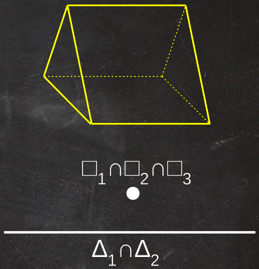

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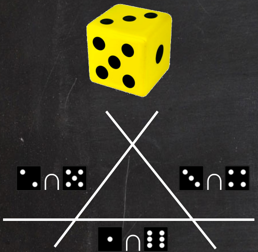

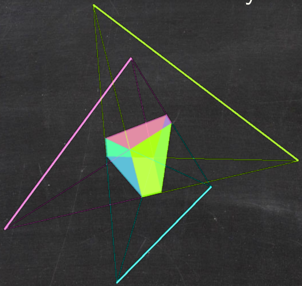

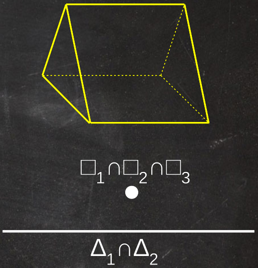

Theorem (K., Ranestad)

If $\mathcal{H}_P$ is simple (i.e. through any point in $\mathbb{P}^n$ pass $\leq n$ hyperplanes),

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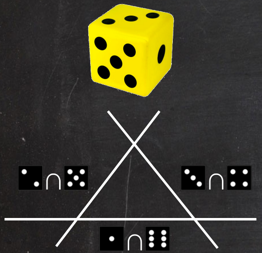

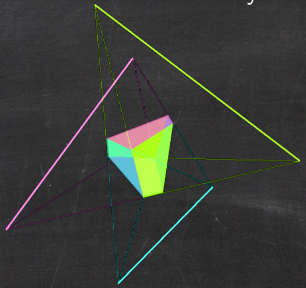

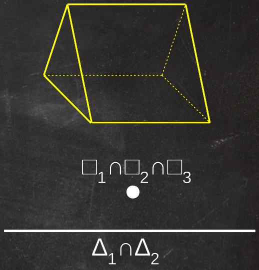

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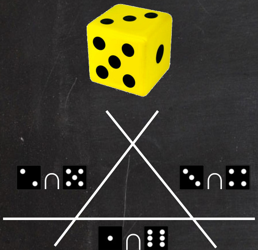

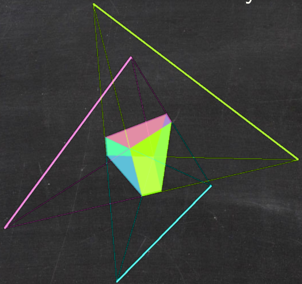

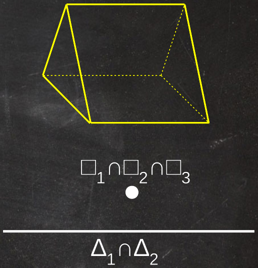

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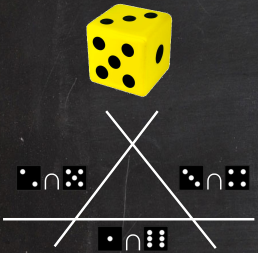

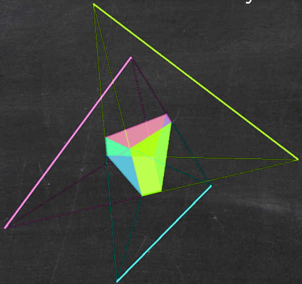

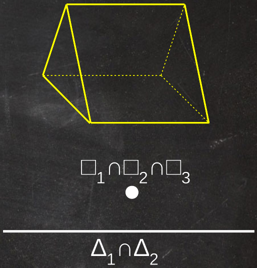

adjoint plane

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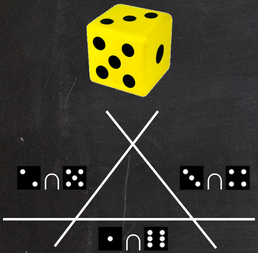
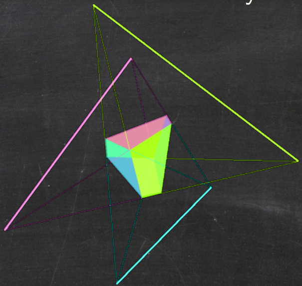
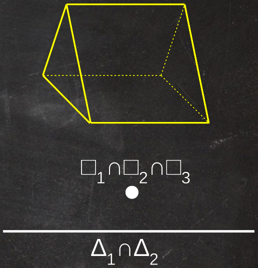




adjoint quadric surface



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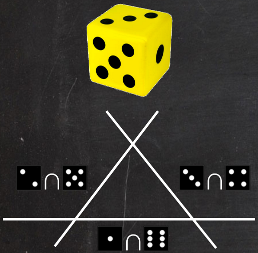
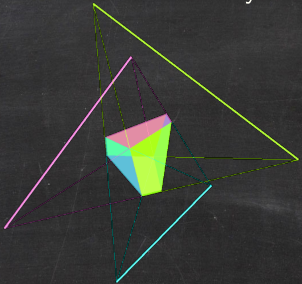


adjoint double plane



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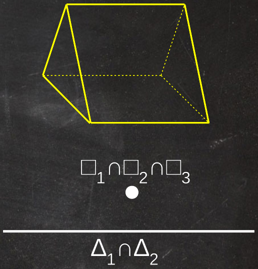

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Proposition (K., Ranestad)

Warren's adjoint polynomial adj_P vanishes along $\mathcal{R}_{P^*}$.
If $\mathcal{H}_{P^*}$ is simple, then $Z(\mathrm{adj}_P) = A_{P^*}$.

Application 1: Segre Classes of Monomial Schemes

Aluffi

- V: smooth variety
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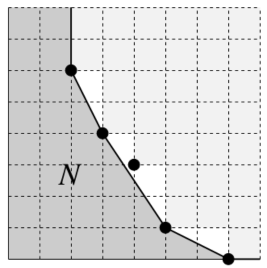
Example: $n = 2$

$\mathcal{A} = \{(2,6), (3,4), (4,3), (5,1), (7,0)\}$



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$$N_{\mathcal{A}} := \mathbb{R}^n_{\geq 0} \setminus \mathrm{convHull}\left(\bigcup_{\mathcal{I} \in \mathcal{A}} (\mathbb{R}^n_{\geq 0} + \mathcal{I}) \right)$$

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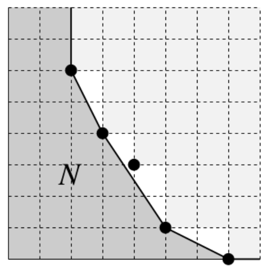

Application 1: Segre Classes of Monomial Schemes

Aluffi

- V: smooth variety
- $X_1, \ldots, X_n$: smooth hypersurfaces meeting with normal crossings in V
- $X^{\mathcal{I}}$: hypersurface obtained by taking X_{i_j} with multiplicity i_j
  for $\mathcal{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$
- $\mathcal{A} \subset \mathbb{Z}^n_{\geq 0}$ defines a **monomial subscheme**

$S_{\mathcal{A}} = \bigcap_{\mathcal{I} \in \mathcal{A}} X^{\mathcal{I}}$ and a Newton region $N_{\mathcal{A}} \subset \mathbb{R}^n_{\geq 0}$

$$N_{\mathcal{A}} := \mathbb{R}^n_{\geq 0} \setminus \mathrm{convHull}\left(\bigcup_{\mathcal{I} \in \mathcal{A}} (\mathbb{R}^n_{>0} + \mathcal{I}) \right)$$

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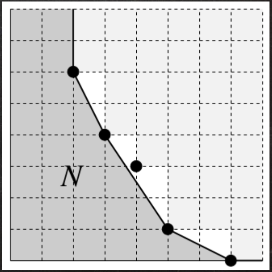

Theorem (Aluffi, (K., Ranestad))

The Segre class of $S_{\mathcal{A}}$ in the Chow ring of V is

$$\frac{n!\, X_1 \cdots X_n\, \mathrm{adj}_{N_{\mathcal{A}}}(-X)}{\prod_{v \in V(N_{\mathcal{A}})} \ell_v(-X)}, \text{ if } N_{\mathcal{A}} \text{ is finite.}$$

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Aluffi

- $N_{\mathcal{A}}$ may have vertices at ∞ in the direction of the standard basis vectors $e_1, \ldots, e_n$

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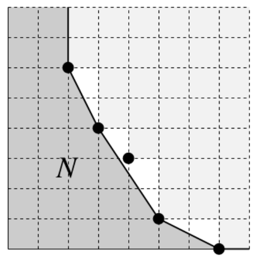

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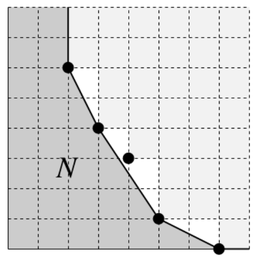

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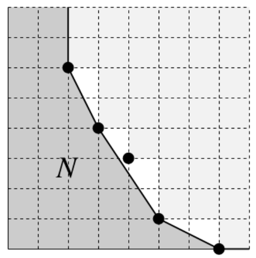

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Example:

$$\frac{2X_1X_2 \operatorname{adj}_{N_{\mathcal{A}}}(-X_1, -X_2)}{X_2(1+2X_1+6X_2)(1+3X_1+4X_2)(1+5X_1+X_2)(1+7X_1)},$$

where

$$\operatorname{adj}_{N_{\mathcal{A}}}(t) = 1 - 15t_1 - 22t_2 + 71t_1^2 + 212t_1t_2 + 95t_2^2 - 105t_1^3 - 476t_1^2t_2 - 511t_1t_2^2 - 84t_2^3.$$

Application 2: Moments of Probability Distributions

K., Shapiro, Sturmfels

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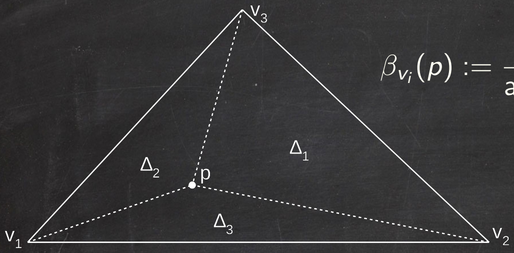
Proposition (K., Shapiro, Sturmfels)

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}} \, m_{\mathcal{I}}(P) \, t^{\mathcal{I}} = \frac{\mathrm{adj_P(t)}}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)},$$

$$\textit{where } c_{\mathcal{I}} := \binom{i_1 + i_2 + \dots + i_n + n}{i_1, i_2, \dots, i_n, n}.$$

Application 3: Barycentric Coordinates



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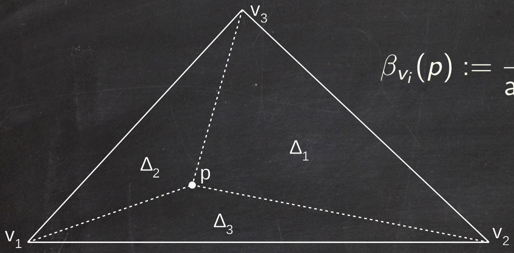

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Definition

Let P be a convex polytope in $\mathbb{R}^n$. A set of functions $\{\beta_u : P^\circ \to \mathbb{R} \mid u \in V(P)\}$ is called **generalized barycentric coordinates** for P if, for all $p \in P^\circ$,

- $\forall u \in V(P) : \beta_u(p) > 0$,
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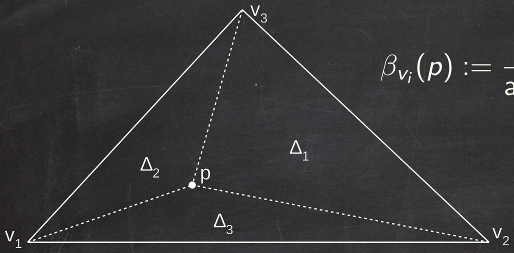

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Barycentric coordinates for simplices are uniquely determined from (i)-(iii).

This is not true for other polytopes!

Application 3: Barycentric Coordinates

Examples of generalized barycentric coordinates for arbitrary polytopes:

- mean value coordinates
- Wachspress coordinates

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Examples of generalized barycentric coordinates for arbitrary polytopes:

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The Wachspress coordinates are the unique generalized barycentric coordinates which are rational functions of minimal degree.

Wachspress Coordinates

Warren (1996)

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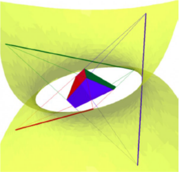
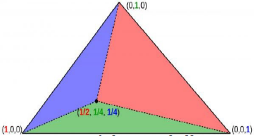
Definition (Warren)

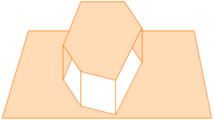
The **Wachspress coordinates** of P are

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\mathrm{adj}_{F_u}(t) \cdot \prod_{F \in \mathcal{F}(P):\, u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}.$$



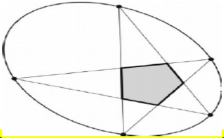


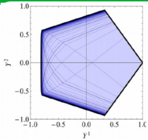
geometric modeling
barycentric coordinates
for arbitrary polytopes



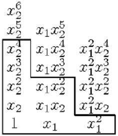
algebraic statistics:
moments of uniform distributions
on polytopes

classical algebraic geometry
adjoint hypersurfaces





physics
scattering amplitudes



intersection theory:
Segre classes of
monomial schemes

X - XII

Why "Adjoint"?

- P: polytope in $\mathbb{P}^n$ with d facets
- $\mathcal{H}_{\mathcal{P}}$: simple hyperplane arrangement spanned by facets of P

Idea:

$P \rightsquigarrow \mathcal{H}_P$

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($D \hookrightarrow \mathbb{P}^n$, $\tilde{D} \hookrightarrow X$)

hypersurface of degree d

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When can we find a polytopal hypersurface D?

Adjunction formula: $K_{\tilde{D}} = (K_X + [\tilde{D}])|_{\tilde{D}}$

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Polytopal Hypersurfaces

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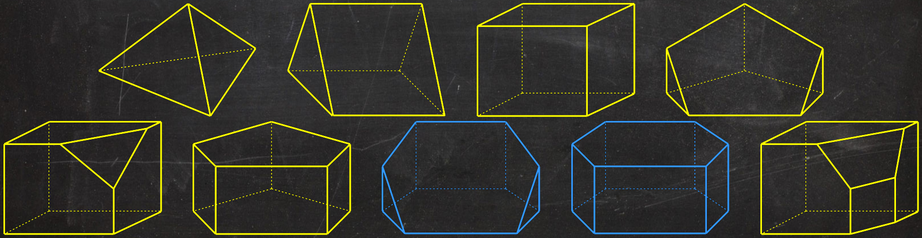
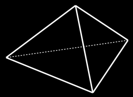
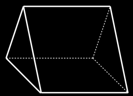
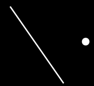
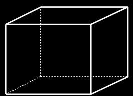
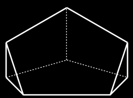
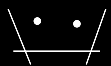
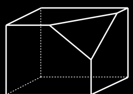
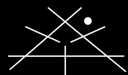
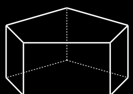
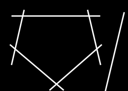
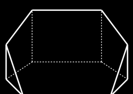
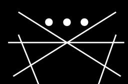
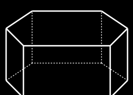
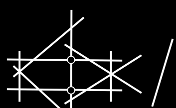
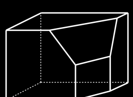
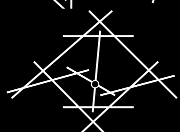
Theorem (K., Ranestad)

Let $\mathcal{C}$ be a combinatorial type of simple polytopes in $\mathbb{P}^3$ and let P be a general polytope of type $\mathcal{C}$. There is a polytopal surface D iff $\mathcal{C}$ is one of:



In that case, the general D is either an elliptic surface or a K3-surface.

comb. type	facet sizes	$\mathcal{R}_P$	(a,b,c)	W_P (deg., sec. genus)	$\overline{w_P(A_P)}$ (deg., sec. genus)	$\dim \Gamma_P$	$\overline{w_P(D)}$ (deg., sec. genus)
	3 3 3 3		$(0,0,0)$	$\mathbb{P}^3$ $(1,0)$	0	34	minimal K3 (smooth quartic in $\mathbb{P}^3$)
	4 4 4 3 3		$(1,0,0)$	$\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ $(3,0)$	line	23	minimal K3 $(8,5)$
	4 4 4 4 4 4		$(0,0,0)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ $(6,1)$	twisted cubic curve	26	minimal K3 $(12,7)$
	5 5 4 4 3 3		$(2,2,0)$	$W_P \subset \mathbb{P}^7$ $(8,3)$	quadric surface $(2,0)$	17	non-minimal K3 $(14,9)$
	5 5 5 4 4 4 3		$(1,6,0)$	$W_P \subset \mathbb{P}^9$ $(15,9)$	del Pezzo surface in $\mathbb{P}^5$ $(5,1)$	7	non-minimal K3 $(19,12)$
	5 5 4 4 4 4 4		$(0,5,0)$	Fano 3-fold in $\mathbb{P}^9$ $(14,8)$	rational scroll in $\mathbb{P}^5$ $(4,0)$	12	non-minimal K3 $(18,11)$
	6 6 4 4 4 3 3		$(3,6,1)$	$W_P \subset \mathbb{P}^9$ $(17,11)$	rational elliptic surface in $\mathbb{P}^5$ $(7,3)$	4	minimal elliptic $(22,15)$
	6 6 4 4 4 4 4 4		$(0,12,2)$	$W_P \subset \mathbb{P}^{11}$ $(27,22)$	elliptic K3-surface in $\mathbb{P}^7$ $(12,7)$	3	minimal elliptic $(26,17)$
	5 5 5 5 4 4 4 4		$(0,16,0)$	$W_P \subset \mathbb{P}^{11}$ $(27,22)$	K3-surface in $\mathbb{P}^7$ $(12,7)$	1	non-minimal K3 $(24,15)$