# understanding Linear Convolutional Neural Networks via sparse factorizations of real polynomials

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joint work with

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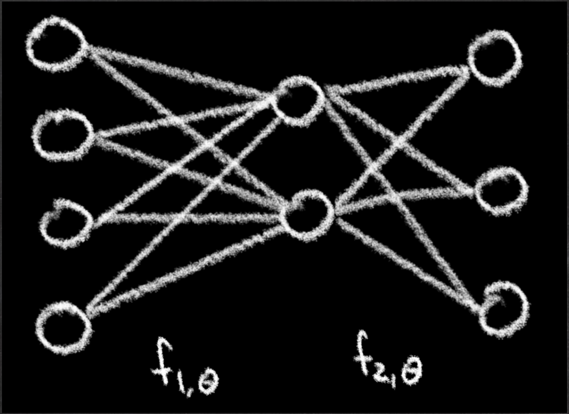
# feedforward neural networks



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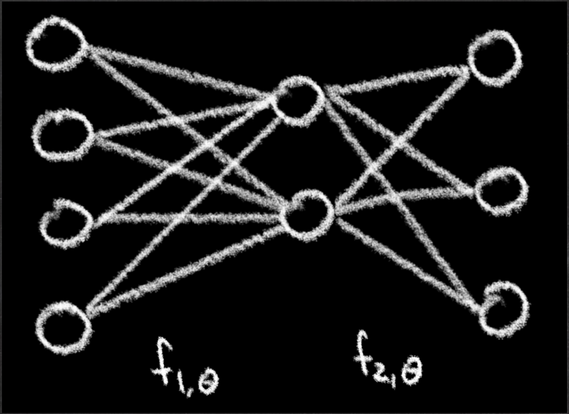


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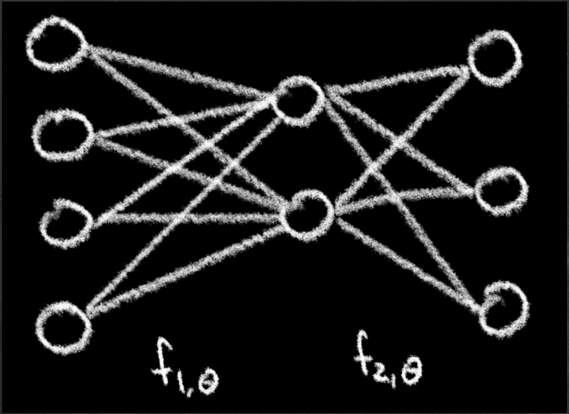


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$\mathcal{M} =$ function space / neuromanifold, $L = \#$ layers

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Given training data $\mathcal{D}$, the goal is to minimize the loss

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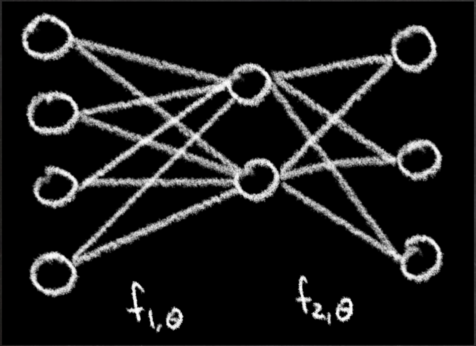
**In this talk:**
What is the impact of changing from dense layers to convolutional layers?

# linear dense networks




In this example:

$$\mu : \mathbb{R}^{2\times 4} \times \mathbb{R}^{3\times 2} \longrightarrow \mathbb{R}^{3\times 4},$$
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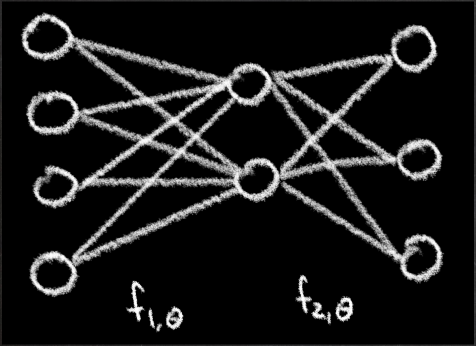


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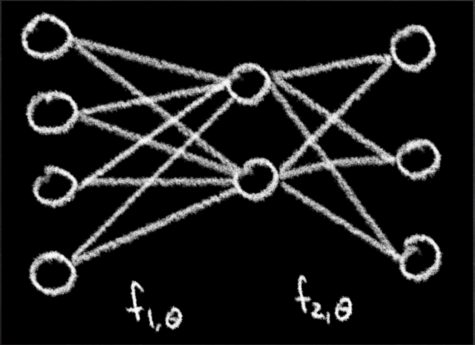


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In general:

$$\mu : \mathbb{R}^{k_1\times k_0} \times \mathbb{R}^{k_2\times k_1} \times \ldots \times \mathbb{R}^{k_L\times k_{L-1}} \longrightarrow \mathbb{R}^{k_L\times k_0},$$
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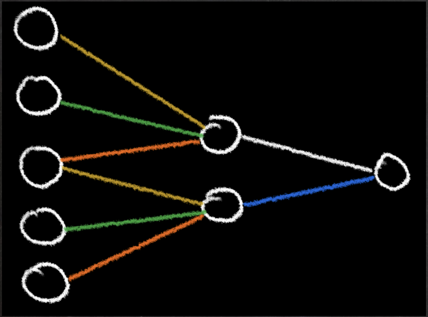
$\mathcal{M} = \{W \in \mathbb{R}^{k_L\times k_0} \mid \operatorname{rank}(W) \leq \min(k_0, \ldots, k_L)\}$ is an algebraic variety and we know its singularities etc.

# Linear Convolutional Networks (LCNs)

with 1D-convolutions



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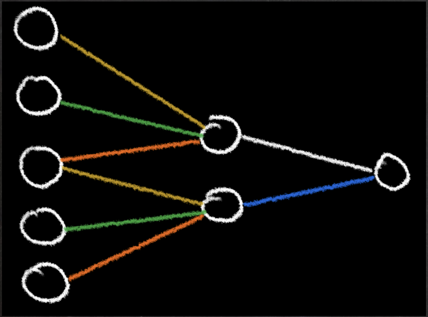

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$$T_{u,2} = \begin{bmatrix} u_0 & u_1 & u_2 & 0 & 0 \\ 0 & 0 & u_0 & u_1 & u_2 \end{bmatrix}$$
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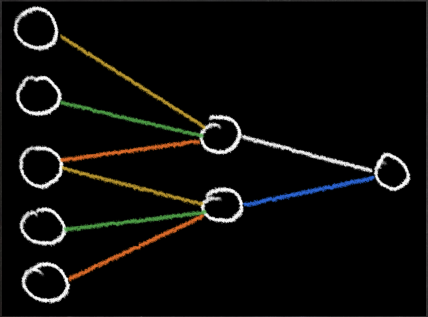

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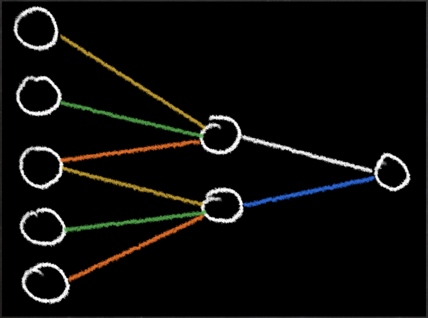

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is a convolutional matrix of stride $s$ with filter $w$

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Hence, we reinterpret $\mu$ as

$$\begin{aligned} \mu : \mathbb{R}[x^{S_1}]_{\leq d_1} \times \ldots \times \mathbb{R}[x^{S_L}]_{\leq d_L} &\longrightarrow \mathbb{R}[x]_{\leq d_1 S_1 + \ldots + d_L S_L}, \\ (P_1, \dots, P_L) &\longmapsto P_L \cdots P_1 \end{aligned}$$

# LCN function spaces

$$\mu : \mathbb{R}[x^{S_1}]_{\leq d_1} \times \ldots \times \mathbb{R}[x^{S_L}]_{\leq d_L} \longrightarrow \mathbb{R}[x]_{\leq d}, \text{where } d := \sum_i d_i S_i$$
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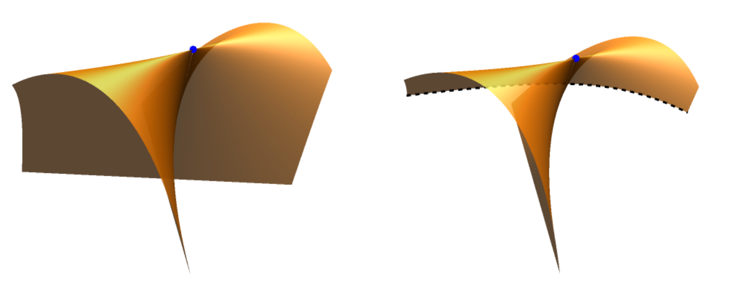
**Theorem:** The function space $\mathcal{M}_{\boldsymbol{d},\boldsymbol{S}} = \mathrm{im}(\mu)$ is a semi-algebraic, Euclidean-closed subset of $\mathbb{R}[x]_{\leq d}$ of dimension $d_1 + \ldots + d_L + 1$.



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**Corollary:** $\mathcal{M}_{\boldsymbol{d},\boldsymbol{S}}$ is full-dimensional in $\mathbb{R}[x]_{\leq d}$ if and only if all strides $s_i = 1$.

# comparison

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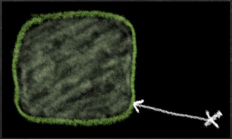

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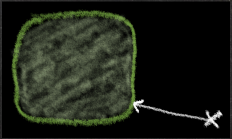

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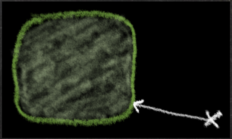

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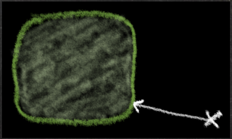

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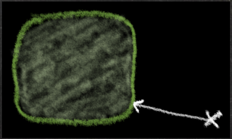

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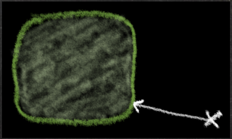

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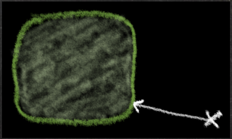

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# training with the squared error loss

Given training data $\mathcal{D} = \{(X_i, Y_i) \in \mathbb{R}^{k_0} \times \mathbb{R}^{k_L} \mid i = 1, \dots, N\}$, the squared error loss on the function space is

$$\ell_{\mathcal{D}} : \mathbb{R}^{k_L \times k_0} \longrightarrow \mathbb{R},$$

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Training an LCN minimizes the squared error loss on the parameter space:

$$\begin{aligned} \mathcal{L}_{\mathcal{D}} : \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_L} &\xrightarrow{\mu} \mathcal{M}_{\boldsymbol{d},\boldsymbol{s}} \subseteq \mathbb{R}^{k_L \times k_0} \xrightarrow{\ell_{\mathcal{D}}} \mathbb{R}, \\ (w_1, \ldots, w_L) &\longmapsto T_{w_L, s_L} \cdots T_{w_1, s_1} \longmapsto \ell_{\mathcal{D}}(T_{w_L, s_L} \cdots T_{w_1, s_1}) \end{aligned}$$

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**Theorem**

Consider an LCN with all strides $> 1$. Let $N \geq \sum_i d_i S_i + 1$.
For almost all data $\mathcal{D} \in (\mathbb{R}^{k_0} \times \mathbb{R}^{k_L})^N$, every critical point $\boldsymbol{w}$ of $\mathcal{L}_{\mathcal{D}}$ satisfies one of the following:

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- $\mu(\boldsymbol{w}) = 0$, or
- $\mu(\boldsymbol{w})$ is a smooth, interior point of $\mathcal{M}_{\boldsymbol{d},\boldsymbol{S}}$.

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A critical point $\theta \in \mathrm{Crit}(\mathcal{L}_\mathcal{D})$ is called **spurious** if $\mu(\theta) \notin \mathrm{Crit}(\ell_\mathcal{D})$.

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| | | full-dimensional | low-dimensional |
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**Theorem**

Consider an LCN with all strides $> 1$. Let $N \geq \sum_i d_i S_i + 1$.
For almost all data $\mathcal{D} \in (\mathbb{R}^{k_0} \times \mathbb{R}^{k_L})^N$, every critical point $\boldsymbol{w}$ of $\mathcal{L}_{\mathcal{D}}$ satisfies one of the following:

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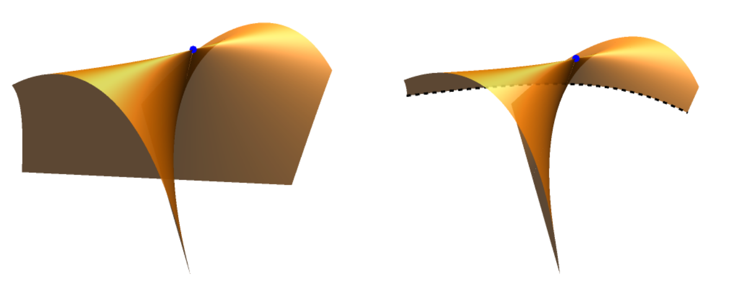
  In particular, $\mu(\boldsymbol{w})$ is a critical point of $\ell_{\mathcal{D}}|_{\mathrm{Reg}(\mathcal{M}^{\circ}_{\boldsymbol{d},\boldsymbol{S}})}$.

# reducing LCNs

$\mu : \mathbb{R}[x]_{\leq 2} \times \mathbb{R}[x^2]_{\leq 1} \to \mathbb{R}[x]_{\leq 4}$ $\qquad$ $\mu : \mathbb{R}[x]_{\leq 1} \times \mathbb{R}[x]_{\leq 1} \times \mathbb{R}[x^2]_{\leq 1} \to \mathbb{R}[x]_{\leq 4}$



$$\begin{array}{ccccc} \mathbb{R}[x]_{\leq 1} \times \mathbb{R}[x]_{\leq 1} & \times & \mathbb{R}[x^2]_{\leq 1} & \longrightarrow & \mathcal{M}_{(1,1,1),(1,1,2)} \\ \downarrow & & \downarrow & & \big\downarrow \\ \mathbb{R}[x]_{\leq 2} & \times & \mathbb{R}[x^2]_{\leq 1} & \longrightarrow & \mathcal{M}_{(2,1),(1,2)} \end{array}$$

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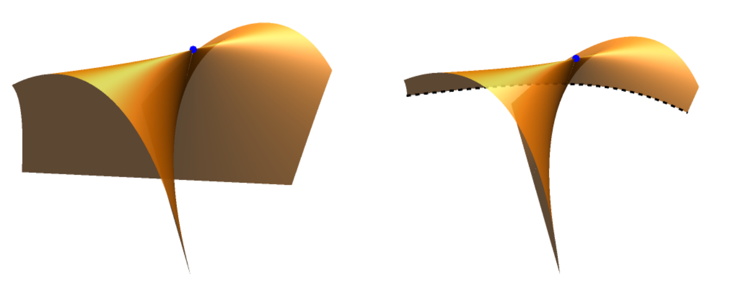

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Given an LCN $(\boldsymbol{d}, \boldsymbol{S})$, merging neighboring layers with the same $S_i$ yields an LCN $(\tilde{\boldsymbol{d}}, \tilde{\boldsymbol{S}})$ with $1 = \tilde{S}_1 < \tilde{S}_2 < \tilde{S}_3 < \ldots$ (i.e., all strides $> 1$), called the reduced LCN.

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**Lemma:** $\mathcal{M}_{\boldsymbol{d},\boldsymbol{S}} \subseteq \mathcal{M}_{\tilde{\boldsymbol{d}},\tilde{\boldsymbol{S}}}$ and $\overline{\mathcal{M}}_{\boldsymbol{d},\boldsymbol{S}} = \overline{\mathcal{M}}_{\tilde{\boldsymbol{d}},\tilde{\boldsymbol{S}}}$,
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# Example

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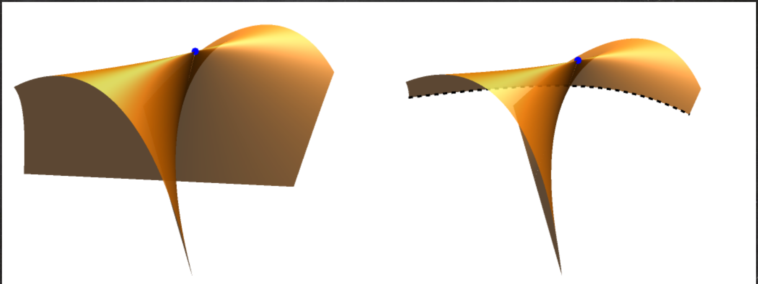

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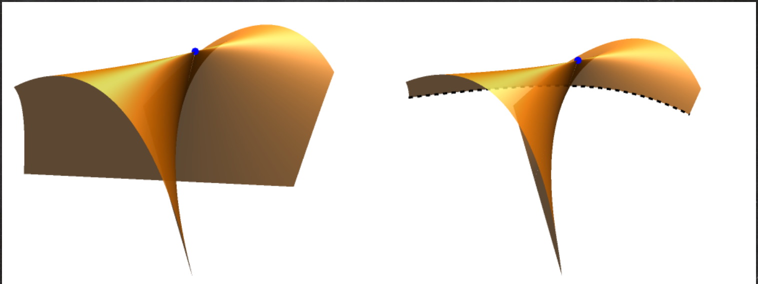

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# Relative Boundary

$\partial \mathcal{M}_{\boldsymbol{d},\boldsymbol{S}}$ = points in $\mathcal{M}_{\boldsymbol{d},\boldsymbol{S}}$ that are limits of sequences in $\overline{\mathcal{M}}_{\boldsymbol{d},\boldsymbol{S}} \setminus \mathcal{M}_{\boldsymbol{d},\boldsymbol{S}}$.

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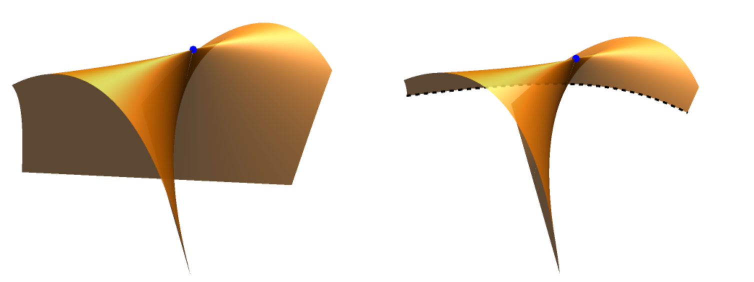

reduced boundary points have at least codimension 2

stride-1 boundary points (if existent) have codimension 1

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